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SELECTED PAPERS

ON

WAVE MECHANICS

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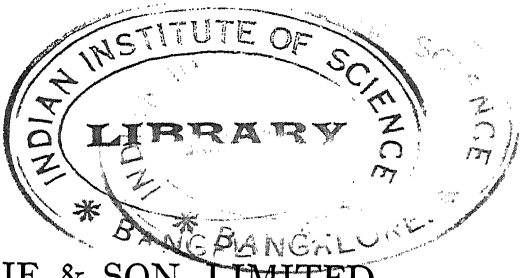

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This translation has been revised by the authors, who have added a few notes. Editorial references to well-known textbooks are enclosed in square brackets.

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CONTENTS

Black Radiation and Light Quanta*

BY

LOUIS DE BROGLIE

(*Le Journal de Physique et le Radium*, Vol. 3 (1922), p. 422)

The object of this paper is to establish a number of known results of the theory of radiation by reasoning dependent on thermodynamics, the kinetic theory, and the theory of quanta alone, without the intervention of electromagnetism.

The hypothesis we adopt is that of light quanta. Black radiation in equilibrium at temperature T is considered as a gas formed of atoms of light of energy $W = h\nu$. In this essay we shall neglect molecules of light with 2, 3, . . ., n atoms $h\nu$; that is to say, we ought to arrive at Wien's law of radiation, for, from the point of view of light quanta, Wien's formula is obtained from Planck's general equation † by neglecting associations of the atoms.

In accordance with the formulæ of relativity mechanics, the mass of the atoms of light is taken as equal to $\frac{h\nu}{c^2}$, the quotient of the energy by the square of the velocity of light. Their momentum is $\frac{h\nu}{c} = \frac{W}{c}$.‡

* This paper was written at the beginning of 1922, two years before the well-known one of S. N. Bose on the light-quanta statistics [*Zeitsch. für Physik*, Vol. **27** (1924), p. 384]; it was the origin of the ideas of the author on wave mechanics.

† [O. W. Richardson, *Electron Theory of Matter*, p. 243; J. H. Jeans, *Report on Radiation and the Quantum Theory*, and *Dynamical Theory of Gases*.]

‡ Relativity dynamics gives $W = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$ for the kinetic energy of a body of proper mass m_0 moving with velocity $v = \beta c$, and $G = \frac{m_0 v}{\sqrt{1-\beta^2}}$ for its momentum. If the ratio β is small, we get back to the results of ordinary mechanics, $W = \frac{m_0v^2}{2}$: $G = m_0v = \frac{2W}{v}$. For the atom of light, however, m_0 must become infinitely small and β infinitely close to unity in such a way that $\frac{m_0}{\sqrt{1-\beta^2}}$ has a definite value m. We then have $W = mc^2$ and $G = mc = \frac{W}{c}$: these are the relationships used in the text.

Let n be the number of atoms of light contained in the unit of volume. On unit surface of the boundary, $\frac{1}{6}nc$ atoms of light impinge per second, each of which has momentum equal to $\frac{W}{c}$. The force on unit surface, or the pressure, is therefore $2 \cdot \frac{1}{6}nc \cdot \frac{W}{c} = \frac{1}{3}nW$: it is equal to one-third of the energy contained in unit volume. This result also follows from the electromagnetic theory and has been verified by experiment.

The number of atoms of light of energy W (i.e. of energy between W and $W + dW$) which are situated in the element of volume $dx\,dy\,dz$, and the components of whose momentum lie between p and $p + dp$, q and $q + dq$, r and $r + dr$, is given by the following formula of statistical mechanics, which still applies,*

$$dn_W = C\,e^{-\frac{W}{kT}}\,dx\,dy\,dz\,dp\,dq\,dr,$$

where C is a constant.

To obtain the total number of atoms of energy W we have to integrate throughout the volume, replace $dp\,dq\,dr$ by $4\pi G^2\,dG$, where G is the length of the vector of momentum, and substitute for G its value $\frac{W}{c}$.

This number of atoms of energy W per unit volume can then be expressed by

$$dn_W = C'\,e^{-\frac{W}{kT}}\,W^2\,dW,$$

where C' is another constant.

Integration for all values of W from zero to infinity should give the number n of atoms of light per unit volume This fixes the value of the constant, and we obtain

$$dn_W = \frac{n}{2k^3T^3}\,e^{-\frac{W}{kT}}\,W^2\,dW.$$

The total energy du of these atoms of energy W is therefore

$$du_W = \frac{n}{2k^3T^3}\,e^{-\frac{W}{kT}}\,W^3\,dW$$

per unit volume.

Let us now attempt to determine n. Let us *assume* that this number is a function of the temperature only: then this function

* In relativity dynamics the equations of motion are always canonical and Liouville's Theorem is always valid.

can be determined thermodynamically. In fact, the total energy per unit volume is

$$\int_0^\infty du_W, \text{ or } 3nkT,$$

for $$\int_0^\infty e^{-\frac{W}{kT}} W^3\, dW = k^4T^4 \int_0^\infty e^{-x} x^3\, dx = 6k^4T^4.$$

This result suggests a remark. Each atom of light possesses, on the average, energy $3kT$ and not $\frac{3}{2}kT$ as in the case of the molecules of an ordinary gas, whose velocities are as a rule small compared with the velocity of light. Thus we come back to a fact which the electromagnetic theory explains by the equality of the electric and magnetic energies of a wave of light. This parallelism is reached by using relativity formulæ, which alone allow the exact value of the pressure of radiation, calculated above, to be obtained from the quantum theory of light, whereas the old corpuscular theory of light leads to a value twice the correct one.

The total energy of the gas is therefore $U = 3nkTV$ and the differential of its entropy is

$$\begin{aligned} dS &= \frac{1}{T}(dU + p\, dV) \\ &= \frac{1}{T}\left(3nkV\, dT + 3nkT\, dV + 3kVT\frac{dn}{dT}dT + nkT\, dV\right), \end{aligned}$$

since the pressure is one-third of the energy per unit volume. Hence

$$dS = \left(\frac{3nkV}{T} + 3kV\frac{dn}{dT}\right) dT + 4nk\, dV.$$

In order to make dS an exact differential, we must have

$$\frac{3nk}{T} + 3k\frac{dn}{dT} = 4k\frac{dn}{dT}, \text{ or } \frac{dn}{dT} = \frac{3n}{T},$$

the solution of which I write in the form $n = Ak^3T^3$, A being a constant unknown for the present. This constant is related to Stefan's constant σ, for the energy per unit volume is

$$3nkT = 3Ak^4T^4,$$

whence, by comparison, $\sigma = 3Ak^4$.

Substituting the value of n in the expression for dS, it becomes

$$dS = 12\, Ak^4T^2\, V\, dT + 4Ak^4T^3\, dV,$$

whence $$S = 4Ak^4T^3V$$

without another constant, since for $T = 0$, $n = 0$, the gas no longer exists.

Since $A = \frac{\sigma}{3k^4}$, we obtain the classical expression $S = \frac{4}{3}\sigma T^3 V$. The free energy $F = U - TS$ can at once be found: it is equal to

$$3nVkT - T \cdot 4nkV = -nVkT = -AVk^4T^4$$

or to $-NkT$, where N is the total number of atoms in the volume V. There is no constant to be added, since the proper mass of the atoms is nil.*

The quantity of energy which atoms of energy W possess per unit volume is

$$du_W = \frac{A}{2} e^{-\frac{W}{kT}} W^3 \, dW,$$

and since $W = h\nu$,

$$du_\nu = \frac{Ah^4}{2} e^{-\frac{h\nu}{kT}} \nu^3 \, d\nu.$$

Thus we obtain the form of Wien's law. Can we calculate the value of the numerical coefficient in this law (without using the experimental value of σ, of course)?

We can attempt to do so by the method which has enabled Planck, Sackur, Tetrode, and others to calculate the "chemical constant".† We shall follow the argument recently developed by Planck.‡ If a gas consists of N atoms at temperature T, the law of canonical distribution which was proposed by Gibbs and which M. Léon Brillouin has put on a solid basis by utilizing the idea of a thermostat, leads to the formula

$$F = -kT \log \sum_n e^{-\frac{\epsilon_n}{kT}}$$

for the free energy, the sum being taken for all possible states of the gas. This sum can be expressed as an integral taken over the whole phase extension of $6N$ dimensions, an integral which is itself equivalent to the product of N sextuple integrals taken over the phase extension of each molecule, if care be taken, as Planck explains in the article mentioned, to divide the result by $N!$. The theory of quanta introduces the hypothesis of an elementary domain of phase extension of size g; g has the dimensions of the cube of an action, and the calculation of the chemical constant leads us to set $g = h^3$ (h is Planck's constant).

The expression for F can then be written

$$F = -kT \log \left[\left(\frac{\iiint\iiint e^{-\frac{W}{kT}} \, dx \, dy \, dz \, dp \, dq \, dr}{g} \right)^N \frac{1}{N!} \right]$$

* The thermodynamic potential $U - TS + pV$ is identically zero.

† [H. S. Taylor, *Treatise on Physical Chemistry*, p. 1137.]

‡ *Annalen der Physik*, Vol. **66** (1921), p. 365.

$$= -kNT \log\left[\frac{eV}{Ng}\int_0^\infty e^{-\frac{W}{kT}} 4\pi G^2\, dG\right]$$

$$= -kNT \log\left[\frac{8\pi eV}{Ng} \times \frac{k^3T^3}{c^3}\right].$$

We found that $F = -NkT$, no constant being added, since the proper mass of atoms of light is negligible. In order to make the two expressions identical we must have $\log\left[\frac{8\pi eV}{Ng}\,\frac{k^3T^3}{c^3}\right] = 1$, whence, since $N = Ak^3T^3V$,

$$A = \frac{8\pi}{c^3 g} = \frac{8\pi}{c^3 h^3}.$$

Consequently du_ν becomes

$$du_\nu = \frac{4\pi h}{c^3} e^{-\frac{h\nu}{kT}} \nu^3\, d\nu.$$

The expression differs from Wien's law by a factor 2. This difference is not due to a mistake in the calculation, but, as M. Léon Brillouin has pointed out to us, it is probably due to the fact that the idea of the polarization of light was not taken account of in the preceding theory. A more complete theory of light quanta should introduce it in some such form as this: with each atom of light there would be associated an internal state of right-handed or left-handed circular polarization, represented by an axial vector in the direction of the velocity of propagation. Two atoms having the same position and the same velocity, if they were to be regarded as identical in the calculation of F, would further require to be polarized in the same sense (right-handedly or left-handedly); this would introduce a factor 2^N under the logarithm sign in the expression for F, thus restoring the exact value of the numerical coefficient of Wien's law.

By considering a mixture of monatomic, diatomic, triatomic, . . . " gases of light ", we should also be able to obtain Planck's law in the form

$$du_\nu = \frac{8\pi h}{c^3} \nu^3 \left[e^{-\frac{h\nu}{kT}} + e^{-\frac{2h\nu}{kT}} + e^{-\frac{3h\nu}{kT}} + \ldots\right]. \; d\nu$$

This would require some rather arbitrary hypotheses, and we shall not proceed further in this direction.

We can also arrive at the conception of a gas of atoms of light in the following way.

Consider a gas formed of N atoms of " proper mass " m_0 in equilibrium at temperature T. Suppose that relativity dynamics applies to these atoms and neglect all interaction between the atoms:

thus our gas is a perfect gas. The energy and momentum are given by the equations:

$$W = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right); \quad \mathbf{G} = \frac{m_0}{\sqrt{1-\beta^2}}\mathbf{v}; \quad \beta = \frac{v}{c}.$$

Statistical mechanics gives the number dN of atoms whose energy lies between W and $W + dW$ (see above),

$$dN_W = CNe^{-\frac{W}{kT}} G^2\, dG$$

$$= CNe^{-\frac{W}{kT}} m_0{}^2c\sqrt{\alpha(\alpha+2)}\,(\alpha+1)\, dW,$$

putting $\frac{W}{m_0c^2} = \alpha$ for short. If the mass m_0 is sufficiently large to make the quotient $\frac{W}{m_0c^2}$ very small for practically all the atoms (and this is what happens in the case of material gases at ordinary temperatures), we revert to Maxwell's ordinary formula. Suppose, on the contrary, that the mass m_0 is very small; then practically all the atoms will have velocities very close to c: it may indeed happen, if m_0 be small enough, that the number of molecules whose velocity differs from c by more than one-millionth is negligible. In that case α will be much greater than unity, and we can write:

$$dN_W = C'Ne^{-\frac{W}{kT}} W^2\, dW,$$

a formula from which, as we have seen, the Planck-Wien law is deduced.

The hypothesis of the quantum theory of light should therefore, with the adoption of relativity dynamics, lead us to regard the atoms of light (supposed of the same very small mass) as moving with velocities which vary according to their energy (frequency), but which are all very close to c. We should thus explain why light appears to be propagated (within the limits of experimental precision) with exactly the velocity which in Einstein's formulæ plays the part of infinite velocity.*

Summing up, the essential conclusions of the present paper are as follows:

1. By means of the quantum theory of light, coupled with the rules of statistical mechanics and of thermodynamics, we can reobtain all the results of the thermodynamics of radiation and even

* The "radiation" of frequency ν would be carried by atoms of mass m_0 displaced with velocity $c - \frac{c^5m_0{}^2}{2h^2\nu^2}$, the quantity $\frac{c^5m_0{}^2}{2h^2\nu^2}$ escaping experimental detection on account of the smallness of m_0.

the Planck-Wien law of distribution.* These results, however, expressly assume that the formulæ of relativity dynamics are used for atoms of light.

2. There is no doubt that there is a close connexion between the chemical constant of gases and Stefan's constant for black radiation. This connexion has been set forth already by M. Lindemann in a recent paper on the vapour pressure of solids.† He reveals to us a new aspect of the constant interaction between matter and radiation.

* On the question of light quanta, see Emden, *Phys. Zeitschr.*, **22** (1921), 509; L. de Broglie, *Comptes Rendus*, **175** (1922), 811.

† *Phil. Mag.*, **39**, pp. 21–5.

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On the Parallelism between the Dynamics of a Material Particle and Geometrical Optics

BY

LOUIS DE BROGLIE

(*Le Journal de Physique et le Radium*, Vol. 7 (1926), p. 1)

Summary.—By associating the propagation of a wave with the motion of a material particle, the energy and momentum of the particle can be related to the frequency and phase velocity of the wave in such a way that the usual equations of dynamics are derived from a dispersion formula.

The corpuscular theory of light comes up against difficulties when propagation in refracting media is studied; one of these difficulties, which is of great historical importance, relates to a supposed contradiction between the principle of least action and Fermat's law. By deducing the dynamics from the theory of waves, we can consider the question from a new point of view and remove certain objections.

1. **Classical Ideas.**—The object of this exposition is to show how the ideas about quanta which I have recently developed permit a precise statement to be made as to the parallelism, the existence of which has been indicated for so long, between the dynamics of a particle and geometrical optics.

Let us commence by recalling some outstanding laws of the theory of waves, *without confining ourselves specially to light waves.*

To begin with, I shall give the following general definition: A physical phenomenon is said to be propagated in simple sinusoidal waves, if its mathematical definition involves a sinusoidal function of the co-ordinates of space and time, which is called the phase, and which possesses the two following properties: 1. At a point in space it has period T and frequency $\nu = \frac{1}{T}$. 2. The different values of the phase are displaced in space along certain lines called the " rays of the wave ", with a velocity V, which is in general a function of the co-ordinates of space and time as well as of the frequency: this velocity V may also depend on the direction of the ray as looked at from a given point.

In all that follows, I shall for simplicity suppose that the medium is isotropic and in a permanent state. In this case the rays of the wave will have an invariable form and the phase velocity V will be a function of the space co-ordinates and of the frequency alone. We shall express this relationship by the equation

$$n = \frac{c}{V} = \phi(x, y, z, \nu),$$

where c is the classical constant of Maxwell's equations. This equation defines n, the index of refraction.

Besides the velocity V we shall introduce a quantity called "group velocity".* This is defined by supposing that we have not to deal with a simple sinusoidal wave, but with a group of simple sinusoidal waves of nearly equal frequencies comprised within a small interval $\nu - \delta\nu$, $\nu + \delta\nu$. On account of the variation of the refractive index with the frequency, the points where the different simple waves are in the same phase move with a velocity U generally different from V, and a familiar piece of reasoning gives:

$$U = \frac{\partial \nu}{\partial \frac{\nu}{V}} = c \frac{\partial \nu}{\partial (n\nu)}.$$

The study of electromagnetic waves in accordance with classical ideas shows that the energy carried by one of these waves in general moves with the group velocity, which is always less than, or at most equal to, the constant c.

To calculate the movements of the wave it is sufficient to know the frequency and the function which determines the values of the refractive index. A principle which, in optics, bears the name of Fermat, the great French physicist and mathematician, tells us, in fact, that if a ray passes through two given points A and B, the time taken for the phase to go from A to B is a minimum, or in other words: if the phase followed a path differing slightly from the actual ray, it would take longer to go from A to B. Thus we must write:

$$\delta \int_A^B \frac{dl}{V} = \frac{1}{c} \delta \int_A^B n \, dl = 0,$$

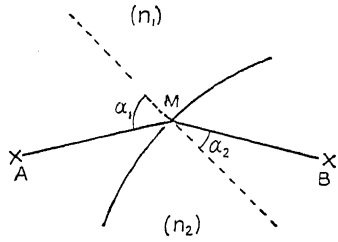
and, since n is known as a function of x, y, z, the path followed by the ray is determined in this way.

Before we come to my own ideas, let us examine two problems of wave propagation which play a great part in geometrical optics.

* [T. H. Havelock, *Propagation of Disturbances in Dispersive Media* (Cambridge Mathematical Tract).]

(*a*) *The passage of a wave from a medium of uniform refractive index* n_1 *to a medium of uniform refractive index* n_2.

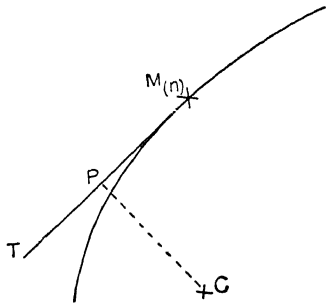
The solution is well known. The ray which goes from the point A of the first medium to the point B of the second is composed of two straight lines which meet at a point M of the surface of separation such that Descartes' law holds:



$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2.$$

(*b*) *The form of the rays in a refracting sphere, the refractive index of which is a function of the distance from the centre and the frequency alone.*

In optics, this problem is called the " problem of astronomical refraction ". The form of the rays is given by an equation,* due to Bouguer, which is deduced from the principle of minimum time. If M is any point on the ray to be found, and if MT is the tangent to the ray at that point, the product of the refractive index at the point M and the distance from the centre of the sphere to the tangent MT has a value which is constant all along the ray:



$$np = C.$$

2. **New Hypotheses.**—Up to this point I have confined myself to observations of a classical nature on waves and their rays. I am now going to introduce a hypothesis which is characteristic of my interpretation of quanta. I shall suppose that there is reason to admit the existence, in a wave, of points where energy is concentrated, of very small corpuscles whose motion is so intimately connected with the displacement of the wave that a knowledge of the laws regulating one of these motions is equivalent to a knowledge of the laws regulating the other.

Conversely, I shall suppose that there is reason to associate wave propagation with the motion of all the kinds of corpuscles whose existence has been revealed to us by experiment.

In other respects, I shall in this paper adopt a point of view slightly different from those which I have developed up to now, *for I shall take the laws of wave propagation as fundamental, and seek to deduce from them, as consequences which are*

* [S. Parkinson, *Treatise on Optics*, § 121.]

valid in certain cases only, the laws of the dynamics of a particle.

I therefore take for granted the principle of minimum time, which follows immediately from undulatory conceptions, and I suppose that the relation which gives the value of the refractive index n, at any point and for any frequency, is known. The motion of the wave being thus determined, it is sufficient, in order to deduce from it the motion of an associated corpuscle, to know the expressions which give the energy W and momentum $\mathbf{g}$ of the wave at any point, as functions of n and ν. For reasons explained in my thesis, one hypothesis is absolutely necessary, namely, to set

$$W = h\nu, \quad g = \frac{h\nu}{V} = \frac{h}{c}(n\nu),$$

the vector $\mathbf{g}$ being tangential to the ray along which the phase at the point considered is propagated. Under these conditions, the corpuscle will follow the ray determined by the principle of minimum time $\delta\int n\,dl = 0$, and its trajectory will, in fact, be that found dynamically by applying Maupertuis' principle $\delta\int g\,dl = 0$.

We shall suppose that the velocity of the moving particle is equal to the group velocity of the waves along the ray, and we shall write

$$v = \beta c = U = c\frac{\partial\nu}{\partial(n\nu)} = \frac{\partial W}{\partial g}.$$

Thus our results are still in agreement with mechanics, for, according to Hamilton's equations,* the velocity is the partial differential coefficient of the energy with respect to the momentum. The preceding hypotheses involve the usual form of the fundamental equation of dynamics, for we have:

$$\frac{dg}{dt} = \left(\frac{\partial g}{\partial l}\right)_W \cdot v = \frac{h}{c}\frac{\partial(n\nu)}{\partial l} \cdot c\frac{\partial\nu}{\partial(n\nu)} = h\frac{\dfrac{\partial(n\nu)}{\partial l}}{\dfrac{\partial(n\nu)}{\partial\nu}} = -h\frac{\partial\nu}{\partial l} = -\frac{\partial W}{\partial l} = F.$$

3. **Dynamics of a Particle.**—We have thus established a close connexion between the propagation of the wave and the dynamics of the associated corpuscle. Let us now see if we can deduce from it the particular relations which dynamics postulates between velocity and mass on the one hand, and between energy and momentum on the other hand.

In order to do this we must in each case specify the form taken by the equation of dispersion $n = \phi(x, y, z, \nu)$. Let us first study the

* [E. T. Whittaker, *Analytical Dynamics*, § 109; M. Born, *Mechanics of the Atom*, p. 20; G. Birtwistle, *Quantum Theory of the Atom*, p. 54.]

propagation of a given type of waves in free space at a great distance from any other matter. The following form of the function ϕ is imposed by the principle of relativity:

$$n = \sqrt{1 - \frac{\nu_0^2}{\nu^2}},$$

where ν_0 is an invariant having the same value for all Galilean systems and characteristic of the intrinsic nature of the wave. A corpuscle associated with the wave in the way just specified will have a velocity

$$v = \beta c = c\frac{\partial \nu}{\partial (n\nu)} = nc.$$

Thus, *in this case*, $n = \beta$, and consequently $V = \frac{c}{n} = \frac{c}{\beta}$, a result which I have proved in many other ways: the energy and the momentum are given by

$$W = h\nu = \frac{h\nu_0}{\sqrt{1-n^2}} = \frac{h\nu_0}{\sqrt{1-\beta^2}}: \quad g = \frac{h}{c}(n\nu) = \frac{W}{c^2}v.$$

These forms can be identified with those of Einstein's dynamics by putting

$$h\nu_0 = m_0 c^2,$$

a relation which defines m_0, the proper mass of the corpuscle, as a function of the invariant ν_0. If the wave considered is a light wave, the invariant ν_0 and consequently the proper mass m_0 must be taken as extraordinarily small: perhaps, to avoid an objection which M. Langevin has kindly pointed out to me, it would even be better boldly to put $\nu_0 = m_0 = 0$. In any case, the velocity of the corpuscle must be extraordinarily close to the constant c, if not equal to it, and the dynamics of the atom of light appears as a limiting case of the dynamics of a material particle. In particular, it is easy to show that this point of view permits of a complete explanation of the various Doppler effects.

Let us now leave the case of free space and consider a medium with spherical symmetry in which the refractive index varies with the distance r from the centre according to the law

$$n^2 = \left(1 - \frac{F(r)}{\nu}\right)^2 - \frac{\nu_0^2}{\nu^2}.$$

The velocity of an associated corpuscle is here found to be given by

$$v = \frac{nc}{1 - \frac{F(r)}{\nu}} = \beta c.$$

The energy and the momentum are given by

$$W = \frac{h\nu_0}{\sqrt{1-\beta^2}} + hF(r); \quad g = \frac{W - hF(r)}{c^2} v.$$

Further, by putting $h\nu_0 = m_0c^2$ and identifying the product $hF(r)$ with the quantity called potential energy in dynamics, we see that the motion of the corpuscle is that of a particle under the action of a central force. The trajectories of the moving body can be determined by Bouguer's equation: $np = C$. Since $g = \frac{h}{c}(n\nu)$, this equation can also be written as

$$gp = C'$$

(where C' is a constant), the equation of areas, which is thus revealed as a particular case of Bouguer's equation.

The most interesting application of the above is the study of the atom of hydrogen as conceived by Bohr. We now regard the atom of hydrogen as a refracting sphere, which, for the waves accompanying the motion of an electron of charge $-e$, possesses a refractive index which varies in accordance with the law stated above, in which we put

$$F(r) = -\frac{eE}{hr},$$

where E is the charge on the nucleus. The rays which return into themselves coincide with the possible trajectories of the electron, and some of them possess the very remarkable property of being in resonance with the wave: it is just these rays which are the " stable " trajectories of Bohr.

Note in passing that the refracting sphere " Bohr-atom " exhibits the phenomenon of mirage.

4. **Optical Dispersion.**—In conclusion, let us investigate the case of a homogeneous material medium in which the refractive index does not depend on the point under consideration but is some function of the frequency. As in the case of free space we can define the proper mass of the associated corpuscle by the invariant equation

$$m_0 = \frac{h}{c^2}\nu\sqrt{1-n^2},$$

but m_0 will be imaginary when n exceeds unity. The energy will have the value $\frac{m_0c^2}{\sqrt{1-n^2}}$, but it will not take the dynamical form $\frac{m_0c^2}{\sqrt{1-\beta^2}}$ unless $n = \beta$, or, according to our general definitions, unless

$n = \frac{dv}{d(nv)}$. This equation has $n = \sqrt{1 + \frac{A}{\nu^2}}$ for its general solution. If the constant A is negative, we obtain a dynamics analogous to that of a material particle in free space. If A is positive, the mass is imaginary and the phase velocity less than c.*

Let us apply these considerations to the quantum of light traversing a medium in which dispersion takes place according to the relation

$$n^2 = 1 + \Sigma_i \frac{\epsilon_i}{\nu_i^2 - \nu^2},$$

which is valid only outside regions of absorption. The quantities ν_i are the proper frequencies of the dispersive medium, and the quantities ϵ_i are its characteristic constants. On the side of each frequency ν_i which is next the red, the proper mass is imaginary and the phase velocity less than c: on the violet side, the proper mass is real and the phase velocity greater than c. Finally, for frequencies much higher than all the ν_i's, the dispersion is given by the equation

$$n^2 = 1 - \frac{\Sigma_i \epsilon_i}{\nu^2},$$

and the dynamics of the quantum becomes analogous to that of a material particle in free space, with the essential difference that the proper mass of the former is determined by the properties of the medium traversed.

5. **Examination of a Classical Piece of Reasoning.**—The upholders of Fresnel's theory raised an objection to the theory of emission which proved fatal to it. These physicists said: " Let us consider the path taken by light passing from a point A situated in a medium of uniform refractive index n_1, to a point B in a medium of uniform refractive index n_2. This path consists of two straight lines which meet at a point of the surface of separation (see first figure on p. 11). Fermat's principle gives (as we have seen)

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2.$$

The corpuscular theory, on the contrary, must start from Maupertuis' principle, $\delta \int_A^B g\, dl = 0$. Since g is the product of the constant mass of the corpuscle of light and the velocity, we shall have $\delta \int_A^B v\, dl = 0$: whence we deduce

$$n_1 \sin \alpha_2 = n_2 \sin \alpha_1.$$

* The velocity of the energy would then be greater than c: so it is physically impossible for this case to occur.

Which of the two formulæ is correct? If the first medium is air or empty space with refractive index unity, and the second medium is water, experiment shows that $\alpha_1 > \alpha_2$. Thus according to the undulatory theory $n_2 > 1$, and the phase is propagated more slowly in water than in air. According to the emission theory, on the contrary, the reverse conclusion holds. Now experiment has decided in favour of the first case: so the corpuscular theory is wrong." Nowadays the above argument fails, because we know that mass varies with velocity; nevertheless, it might seem possible, at first sight, to reinstate it. According to Einstein, in fact, the free particle has momentum $\frac{W}{c^2}v$, where W is its energy. Now the passage from one medium to another cannot make this energy vary. For me, in particular, this conclusion is essential, since a change of energy would be accompanied by a change of frequency. Thus the mass W/c^2 of the moving corpuscle no longer varies, and we must write

$$\delta\int_A^B v\,dl = 0.$$

The objection thus appears to remain in force.

We can escape from it, however, by adopting the point of view developed here, according to which the equations of dynamics are deduced as particular cases of the theory of waves. In my opinion, in fact, we must always return to the definition of g:

$$g = \frac{h}{c}(n\nu) = \frac{W}{c^2}nc.$$

Then the momentum is equal to $\frac{W}{c^2}v$ only if

$$n = \beta = \sqrt{1 + \frac{A}{\nu^2}},$$

and this condition is not in general realized for refracting media under the usual conditions.

The only correct and general form of Maupertuis' principle is therefore:

$$\delta\int_A^B \frac{h\nu}{c}n\,dl = 0.$$

This cannot contradict the undulatory theory, since it is identical with Fermat's principle.

Note in Proof.

If we suppose that the quantum of light has potential energy P in a refracting medium, we are evidently led to write:

$$g = \frac{h\nu - P}{c^2} v = \frac{h\nu}{V},$$

whence we obtain

$$P = h\nu \left(1 - n \frac{d(n\nu)}{d\nu}\right) = h\nu \left[1 - \frac{d(n^2\nu^2)}{d(\nu^2)}\right].$$

If there is a single critical frequency ν_i in the medium the formula of Lorentz gives, outside the zones of absorption,

$$P = -h\nu \frac{\epsilon_i \nu_i^2}{(\nu_i^2 - \nu^2)^2} < 0.$$

Thus everything takes place as if the molecules of the medium *attracted* the quantum, and that all the more strongly as resonance is more closely approached.

The New Atomic Mechanics

BY

LÉON BRILLOUIN

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Summary.—This article explains the methods of the new atomic mechanics, as developed by Heisenberg, *Zts. f. Phys.*, Vol. **33** (1925), p. 879; Born and Jordan, *Zts. f. Phys.*, Vol. **34** (1925), p. 858; Dirac, *Proc. Roy. Soc.*, A, Vol. **109** (1925), p. 642; Kramers, *Physica*, Vol. **5** (1925), p. 369; Born, Jordan, and Heisenberg, *Zts. f. Phys.*, Vol. **35** (1926), p. 557; L. Brillouin, *C. R.*, Vol. **182** (1926), p. 374; Dirac, *Proc. Roy. Soc.*, A, Vol. **110** (1926), p. 561; Born and Wiener, *Zts. f. Phys.*, Vol. **36** (1926), p. 174; K. Lanczos, *Zts. f. Phys.*, Vol. **35** (1926), p. 812, and Vol. **36** (1926), p. 401; N. Bohr, *Naturwiss.* (Jan., 1926); F. London, *Zts. f. Phys.*, Vol. **36** (1926), p. 775; A. Wintner, *Zts. f. Phys.*, Vol. **36** (1926), p. 778.

The author has tried to set forth the essential points of these attempts and to reconcile the often widely differing points of view adopted by their authors, particularly Born and Dirac. Some criticisms as to detail are also stated, notably on the following points: the limitation of matrices to terms with positive indices is arbitrary and has little justification; if negative indices are admitted, new solutions of the problem of the linear oscillator are obtained (§ 6). Matrices are bounded in general, contrary to Born's statement, which depends on the use of an ill-chosen criterion (§ 13).

1. **Introduction.**—The theory of atomic structure as developed in the last dozen years is a somewhat incongruous mixture of classical mechanics and quantum hypotheses. We start from the ordinary equations of motion; we then assume that certain possible trajectories of the electron possess special stability, namely, those for which the integrals of action are equal to an integral multiple of h:

$$\mathcal{J}_r = (\int) p_r \, dq_r = n_r \, h^* . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This first condition is easily admitted: we can conceive the necessity for adding a limitation of this kind to the general laws of mechanics. Bohr's second condition is, however, much more disconcerting: it goes on to state that the observed frequencies do not correspond to those of the privileged motions, but are given by the rule

$$h\nu_{nm} = W_n - W_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* [M. Born, *Mechanics of the Atom*, §§ 2, 9, 10; G. Birtwistle, *Quantum Theory of the Atom*, §§ 27, 32, 45.]

where the W's represent the levels of energy of the different quantized motions.

The important successes of these theories of Bohr and the great part played by them must not cause us to forget the insecurity of their starting-point. The different principles (of selection,* of correspondence,† &c.), which it has been necessary to add afterwards, confirm us in the opinion that the mechanics of quanta is yet to be discovered and that Bohr's hypotheses are only an ingenious first approximation.‡

One problem, namely the investigation of the laws of dispersion, has made the difficulty stand out clearly. Classical theory explained anomalous dispersion round absorption lines very simply, while it seemed impossible to account for it by means of quanta. All such attempts (by Epstein, Born, and others) ended with the disconcerting result that anomalous dispersion should be produced for the real frequencies of motion and not for the absorption frequencies given by the relation (2). This difficult problem has been attacked again by Kramers: he follows an entirely different method. He gives up using the conditions (1), but seeks to rewrite the main formulæ, by replacing the classical frequencies by their quantum values, and letting differential coefficients of the type $\frac{\partial F}{\partial \breve{J}_r}$ appear, which may be replaced by finite differences $F(n,h) - F(m,h)$. This in short means abandoning the conditions (1), which have been recognized as too narrow and arbitrary, in order to be guided by Bohr's principle of correspondence alone.§ The results obtained are very interesting: Kramers and Heisenberg give an expression for the electric moment of the atom under the influence of an incident wave, and this formula accounts for dispersion, scattering (Rayleigh's opalescence), quantum absorption and re-emission, and Wood's resonance.

Encouraged by this success, Heisenberg‖ attempted to correct the initial statements and so to formulate a new atomic mechanics. He remarks that it is illogical to introduce into the equations quanti-

* [M. Born, *loc. cit.*, pp. 138, 152, 180.]

† [M. Born, *loc. cit.*, pp. 8, 9, 60, 99, 220; G. Birtwistle, *loc. cit.*, p. 33; N. Bohr, *Theory of Spectra and Atomic Constitution*, pp. 27, 50, 81.]

‡ With regard to these primitive theories, the following may be consulted: L. Brillouin, *L'atome de Bohr* (Conférences-rapports), and the articles by L. Bloch (*J. Phys.*, Vol. **3** (1922), p. 110) and Boll and Salomon (*J. Phys.*, Vol. **4** (1923), p. 310).

§ Bohr, Kramers, and Slater, *Zts. f. Phys.*, Vol. **24** (1924), p. 69; *Phil. Mag.*, Vol. **47** (1924), p. 785; H. A. Kramers, *Nature*, Vol. **113** (1924), p. 673; Vol. **114** (1924), p. 310; H. A. Kramers and W. Heisenberg, *Zts. f. Phys.*, Vol. **31** (1925), pp. 681–708; van Vleck, *Phys. Rev.*, Vol. **24** (1924), p. 344.

‖ W. Heisenberg, *Zts. f. Phys.*, Vol. **33** (1925), p. 879.

ties which cannot be physically observed, but that in accordance with the principle of correspondence we must seek to replace the quantities furnished by classical mechanics by expansions in which only quantized frequencies occur. For each value W of the total energy, rational mechanics gives a periodic motion which may be expressed as a Fourier series; we can write

$$q_w = \sum_r q_{w,r}\, e^{i2\pi r\nu_w t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where q_w is real and we sum with respect to r from $-\infty$ to $+\infty$, assuming that $q_{w,-r}$ is the conjugate imaginary of $q_{w,r}$,

$$q_{w,-r} = q^*_{w,r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

In future we shall always use asterisks to denote conjugate imaginaries. In atomic mechanics the co-ordinate q appears as an array of terms

$$q = \{q_{nm}\, e^{i2\pi\nu_{nm}t}\} \qquad n, m = 0, 1, 2, 3, \ldots, \quad . \quad (4)$$

corresponding to all possible transitions from a level n to a level m. When an atom is in the state $W_n = nh$, we know that it radiates all the frequencies corresponding to its return to states of less energy; the expression for the co-ordinate q of an atom in state n should therefore include the sum of all these terms; putting $m = n - r$ we have

$$q_n = \sum_r (q_{n,n-r}\, e^{i2\pi\nu_{n,n-r}t} + q_{n-r,n}\, e^{i2\pi\nu_{n-r,n}t}). \quad . \quad . \quad (5)$$

This development corresponds with the expression as a Fourier series (3), and gives a real expression if we take

$$q_{n,m} = q^*_{m,n}, \text{ since } \nu_{n-r,n} = -\nu_{n,n-r}. \quad . \quad . \quad (5a)$$

Heisenberg's general idea is that we must not cling to the particular expansion (5) relating to a special state n, but that we must consider the array of all the terms which can appear in the spectrum, that is, the whole table (4). In order to reason with quantities like (4), we must define the elementary operations of addition and multiplication. Addition will take place term by term: as for multiplication, the rule must be such as to give us an expansion of type (4) with the same frequencies ν_{nm}. Let us therefore put

$$\mathrm{q} = \{q_{nm}\, e^{i2\pi\nu_{nm}t}\}, \quad \mathrm{p} = \{p_{nm}\, e^{i2\pi\nu_{nm}t}\}.$$

If we take a term (n, k) from one side and a term (k, m) from the other, their product will possess a frequency

$$\nu_{nk} + \nu_{km} = \nu_{nm}$$

in virtue of the relations (2). We can then define multiplication

in two different ways, according to the order of the terms:

$$(pq)_{nm} = \sum_k p_{nk} q_{km} \text{ and } (qp)_{nm} = \sum_k q_{nk} p_{km}. \quad . \quad . \quad (6)$$

Heisenberg developed these observations and showed that the equations of mechanics could be satisfied by supposing that the momenta and co-ordinates were represented by expressions like (4). The essential observation was, however, due to Born, who noticed the identity of the formulæ (6) with those for the multiplication of matrices (or determinants). This was the key to the problem, and it enabled a logical presentation of Heisenberg's attempts to be made. In the following paragraphs we shall summarize the extremely important memoirs which have been devoted to this investigation by Born, Heisenberg, and Jordan.*

2. **The Calculus of Matrices. Definitions and Elementary Operations.**—A table of numbers, arranged in rows and columns and depending on two indices n and m,

$$\mathfrak{a} = \{a_{nm}\} = \begin{Bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots\cdots \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Bmatrix}, \quad (7)$$

is called a matrix a.†

These quantities are introduced in Mathematics in the study of systems of linear equations, of bilinear forms, &c. The determinant of a square matrix is a well-known algebraic operation, carried out on the terms of the matrix. We shall only consider infinite matrices, in which n and m can increase without limit.

The equality of matrices is written down term by term, as is also their addition:

$$\mathfrak{a} = \mathfrak{b} \text{ means } a_{nm} = b_{nm}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{c} \text{ means } a_{nm} + b_{nm} = c_{nm}. \quad . \quad . \quad . \quad (9)$$

Multiplication is defined by the sum of the products of corresponding terms taken from the row of the first matrix and the column of the second:

$$\mathfrak{ab} = \mathfrak{c} \text{ means } \sum_k a_{nk} b_{km} = c_{nm}. \quad . \quad . \quad . \quad (10)$$

The associative law holds: the distributive law is applicable only to addition combined with multiplication:

$$\left.\begin{aligned} &\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}; \quad (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}), \\ &(\mathfrak{a} \cdot \mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b} \cdot \mathfrak{c}), \quad \mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{ab} + \mathfrak{ac}. \end{aligned}\right\} \quad . \quad (11)$$

* (B. I) M. Born and P. Jordan, *Zts. f. Phys.*, Vol. **34** (1925), pp. 858–88. (B. II) M. Born, W. Heisenberg, and P. Jordan, *Zts. f. Phys.*, Vol. **35** (1926), pp. 557–615. In what follows we shall refer to these articles as B. I. B. II.

† [See references on p. 73.]

In general, however, ab is not equal to ba: if this condition holds, the matrices a and b are said to be interchangeable or commutable.

The power a^p is defined by repeated multiplication; the matrices a^p, a^q are interchangeable:

$$a^p \cdot a^q = a^q \cdot a^p = a^{p+q}.$$

A matrix in which all the elements are zero, except those in the leading diagonal, which are equal to 1, is called a *unit matrix*: we shall write

$$\mathbf{1} = \{\delta_{nm}\} \text{ where } \delta_{nm} = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } n = m \end{cases}. \quad . \quad . \quad (12)$$

The relations

$$a \cdot \mathbf{1} = \mathbf{1} \cdot a = a \quad . \quad . \quad . \quad . \quad . \quad (13)$$

may be verified immediately.

After defining positive powers of a, we must do so for negative powers. If a^{-1} is a matrix such that

$$a^{-1} \cdot a = a \cdot a^{-1} = \mathbf{1}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we shall call a^{-1} the *inverse matrix* of a.

Let us give some examples in order to make the meaning of these relations quite clear. If a is a limited matrix $[0 \leqslant n \leqslant N, 0 \leqslant m \leqslant N]$, the elements of the inverse matrix are $\frac{A_{mn}}{|a|}$, where A_{nm} is the minor of a_{nm} and $|a|$ the determinant formed by the a_{nm}'s. This result can be generalized for infinite matrices whose determinant $|a|$ is bounded. A *diagonal matrix* $d = \{d_{nn}\}$ has all its elements zero except those for which the two indices are equal; its inverse matrix is also a diagonal matrix and has $\frac{1}{d_{nn}}$ for its elements:

$$d^{-1} = \left\{\frac{1}{d_{nn}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Another simple example is a *matrix of permutation*. Let $k_1 k_2 k_3 \ldots$ be a permutation of the successive integers ($k_n \neq k_m$ if $n \neq m$). Let us form a matrix

$$p = \{p_{nm}\} \text{ with } p_{nm} = \begin{cases} 1 \text{ if } m = k_n \\ 0 \text{ otherwise} \end{cases}. \quad . \quad . \quad (16)$$

The transposed matrix $\tilde{p}$ is got by interchanging the indices:

$$\tilde{p}_{nm} = p_{mn}, \text{ so that } \tilde{p}_{nm} = \begin{cases} 1 \text{ if } n = k_m \\ 0 \text{ otherwise.} \end{cases}$$

By multiplication we obtain

$$p\tilde{p} = \{\Sigma_k p_{nk} \tilde{p}_{km}\} = \{\delta_{nm}\} = \mathbf{1}. \quad . \quad . \quad . \quad (17)$$

The transposed matrix $\tilde{p}$ is therefore equal to p^{-1}. The matrices p and $\tilde{p}$ enable us to permute the rows or the columns of a given matrix: in fact the operation

$$pa = \{\sum_k p_{nk}\, a_{km}\} = \{a_{k_n m}\} \quad . \quad . \quad . \quad . \quad (18)$$

gives a matrix made up of the elements a but with the rows permuted. The product $a\tilde{p} = ap^{-1}$ effects permutation of the columns; the simultaneous permutation of rows and columns may be written

$$a' = p \cdot a \cdot p^{-1}. \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

It is important to note that the product of two matrices can vanish without either of the matrices being zero:

$$a \cdot b = 0 \text{ with } a \neq 0 \text{ and } b \neq 0. \quad . \quad . \quad . \quad (19)$$

This happens, for example, if all the terms a_{nm} vanish except *one*, $a_{n_0 k_0}$, and if all the terms of the row k_0 vanish in b; $b_{k_0 m} = 0$, for $m = 0, 1, 2, \ldots$.

In order to verify that all the terms of ab vanish it is sufficient to apply formula (10).

If we consider a product of matrices $a \cdot b \cdot c \cdot d$, its inverse is $d^{-1}c^{-1}b^{-1}a^{-1}$; indeed, if we form the product $abcd\, d^{-1}c^{-1}b^{-1}a^{-1}$, we can suppress $d\, d^{-1}$, the product of which is **1**; we then have $c\, c^{-1}$ side by side and this is eliminated also, and we finally obtain

$$abcd\, d^{-1}c^{-1}b^{-1}a^{-1} = \mathbf{1}. \quad . \quad . \quad . \quad . \quad (20)$$

3. **Functions of Matrices.**—A *function of matrices* will be defined by an expansion in power series: if $x_1, x_2, x_3, \ldots x_n$ are n matrices, we shall be able to form a function $F(x_1, x_2, \ldots x_n)$ by means of an expansion

$$F(x_1, x_2, \ldots, x_n) = \sum A_{k_1 k_2 \ldots k_n l_1 l_2 \ldots l_n m_1 m_2 \ldots m_n \ldots} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n} x_1^{m_1} x_2^{m_2} \ldots \quad . \quad (21)$$

As changing the order of the factors of a product is not permissible in general, the term just written down cannot be reduced to the form

$$x_1^{k_1+l_1+m_1+\cdots} x_2^{k_2+l_2+m_2+\cdots} \ldots x_n^{k_n+l_n+m_n+\cdots};$$

thus, if we start from a function of ordinary variables $F(x_1, x_2, \ldots, x_n)$ and wish to form an analogous function in the theory of matrices, we can do so in an infinite number of ways. We can restrict this generality by finding a *symmetrized function* F. For each term $A_{p_1 p_2 \ldots p_n} x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ of the expansion of the function F, we shall make a corresponding term in F; the latter will consist of the coefficient $A_{p_1 p_2 \ldots p_n}$ multiplied by the mean of all the products that

can be formed when, for each variable x_i, we have the sum of the exponents

$$k_i + l_i + m_i + \ldots = p_i, \quad \ldots \quad (22)$$

i.e. equal to the original exponent of x_i in F.

The different elements of a matrix a may be functions of a parameter t; the derivative of a with respect to t will be the matrix formed by the derived elements, and we shall write

$$\dot{a} = \{\dot{a}_{nm}\}. \quad \ldots \quad (23)$$

We must investigate the form taken by the derivatives with respect to t of a function of matrices; we shall suppose that the matrices $x_1, x_2, \ldots, x_n$ are made up of elements which are functions of t.

We obviously have

$$\frac{d}{dt}(x_1 + x_2) = \dot{x}_1 + \dot{x}_2: \quad \ldots \quad (23a)$$

but, for a product, the rules obtained are not identical with those of ordinary algebraic calculation: the order of the factors must always be attended to.

The relation

$$\frac{d}{dt}\Big[\sum_k a_{nk}\, b_{km}\Big] = \sum_k \dot{a}_{nk}\, b_{km} + \sum_k a_{nk}\, \dot{b}_{km}$$

becomes, in the case of matrices,

$$\frac{d}{dt}[a \cdot b] = \dot{a}\cdot b + a \cdot \dot{b}. \quad \ldots \quad (24)$$

By repeating the process we obtain

$$\frac{d}{dt}[x_1 x_2 x_3 \ldots x_n] = \dot{x}_1 x_2 x_3 \ldots x_n + x_1 \dot{x}_2 x_3 \ldots x_n + x_1 x_2 \dot{x}_3 \ldots x_n + \ldots + x_1 x_2 x_3 \ldots x_{n-1} \dot{x}_n. \quad (25)$$

In particular, let us consider the case of a power

$$f = x^n$$

$$\dot{f} = \dot{x}x^{n-1} + x\dot{x}x^{n-2} + x^2\dot{x}x^{n-3} + \ldots + x^{n-2}\dot{x}x + x^{n-1}\dot{x}. \quad (26)$$

In ordinary algebra we should regroup all these terms and we should put $\dot{f}$ in the form $\frac{\partial f}{\partial x}\frac{dx}{dt}$; this operation is impossible here, for $\dot{x}$ is not in general commutable with x. Thus I have no logical means of defining the derivative of a matrix function f(x) with respect to x. Similarly, in the case of a function F of several matrices $x_1, x_2, \ldots, x_n$,

we cannot write down a total differential, nor speak of partial derivatives of F, in the ordinary sense of these terms.

We shall see, however, that the quantum mechanics gives rise to an operation very like differentiation, which may be defined in the following way: *

$$\frac{\partial F}{\partial x_1} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [F(x_1 + \epsilon \mathbf{1}, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)]. \quad (27)$$

This operation is algebraically identical with classical differentiation; from x_1^n, we obtain nx_1^{n-1}; from $\sqrt{\phi(x_1)}$, we obtain $\frac{1}{2}\phi^{-\frac{1}{2}}\frac{\partial \phi}{\partial x_1}$, &c.

We shall conclude these generalities by indicating a fundamental property of equations involving matrices. If we have a relation

$$f(x_1, x_2, \ldots, x_n) = 0$$

between different matrices $x_1 \ldots x_n$, the relation remains true if the x_i's are replaced by new matrices

$$X_i = S \cdot x_i \cdot S^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This *canonical transformation* will be of considerable importance in what follows; S, the matrix of the transformation, may be any matrix whatever.

I shall show, in fact, that the function f is simply transformed into

$$F = f(X_1, X_2, \ldots, X_n) = S f(x_1, \ldots, x_n) S^{-1}. \quad . \quad (29)$$

The function f is defined by an expansion in a series of products; the relation (29) will therefore hold for every function if we prove it for a sum and for a product. Now, this follows at once:

$$\left.\begin{aligned} X_1 + X_2 &= Sx_1S^{-1} + Sx_2S^{-1} = S(x_1 + x_2)S^{-1} \\ X_1X_2 &= Sx_1S^{-1}\,Sx_2S^{-1} = Sx_1x_2S^{-1} \text{ since } S^{-1}S = \mathbf{1}. \end{aligned}\right\} \quad (29a)$$

Some little way back we said that the fundamental equations of

* Born gave two definitions, one after the other (B. I, p. 62, and B. II, p. 561); to me the second seems the clearer.

London has observed that these definitions can be stated in the following general form:

The matrix F is made up of the coefficients F_{ik}, and the matrix $q = \{q_{lm}\}$.

The most general derivative is $\frac{\partial F_{ik}}{\partial q_{lm}}$, and it forms a matrix with four indices, i, k, l, m.

Matrices with two indices may be deduced from it by contraction †; Born's first and second definitions correspond to the two following types of contraction:

$$(1) \sum_{ik} \frac{\partial F_{ik}}{\partial q_{lm}} \delta_{ik}, \quad (2) \sum_{lm} \frac{\partial F_{ik}}{\partial q_{lm}} \delta_{lm}.$$

† [T. Levi-Civita, *Absolute Differential Calculus*, p. 77; A. S. Eddington, *Mathematical Theory of Relativity*, § 24.]

the new mechanics will be expressed in finite terms: they will therefore be invariant with respect to a canonical transformation (29).

Let us observe at once that the particular quantities about which we are reasoning are defined, in short, by the fundamental equations (11), (13), and (14), that is, by the statement of the arithmetical rules and the hypothesis that an inverse function exists. Matrices satisfy these rules, but possibly other types of *symbols* do so as well. These rules represent the essentials of the new theory, of which matrices are only a particular example; all the further definitions given in this section and in those that follow can equally well be applied to symbols which satisfy the above rules.

The present attempts to generalize are based on this observation.

4. Atomic Mechanics; the case of Periodic Problems with One Degree of Freedom.

The enunciations have been given for the problems which lead to periodic motion. In the introduction (§ 1) we recalled the fact that in this case a quantized variable q (or its momentum p) must be considered as including all the terms of a matrix

$$\mathrm{q} = \{q_{nm}\, e^{i2\pi\nu_{nm}t}\};\quad \mathrm{p} = \{p_{nm}\, e^{i2\pi\nu_{nm}t}\}. \quad . \quad . \quad (30)$$

The frequencies ν_{nm} are defined by relations of the form (2), and the levels of energy W will be deduced from the fundamental equations. We again suppose that

$$q_{mn} = q^*_{nm} \text{ and } p_{mn} = p^*_{nm}, \quad . \quad . \quad . \quad . \quad (31)$$

where the asterisk serves to denote the conjugate imaginary; matrices of this type are called Hermite matrices.*

The choice of the Hamiltonian function H(p, q) involves a problem in mechanics; and we shall choose the symmetrized function which most closely resembles the classical function. If we are dealing, for example, with the motion of a material particle m under the influence of a potential $V(q)$, we shall write

$$\mathrm{H}(\mathrm{p}, \mathrm{q}) = \mathrm{V}(\mathrm{q}) + \frac{1}{2m}\mathrm{p}^2. \quad . \quad . \quad . \quad . \quad (32)$$

* The imaginary parts of two terms q_{nm} and q_{mn}, which are symmetrical with respect to the diagonal, are equal and of opposite sign; the diagonal terms are real.

An arbitrary function of different Hermite matrices is not necessarily a Hermite matrix; it will easily be verified that p + q and pq + qp are of the Hermite type, whereas pq and qp are not so unless p and q are commutable; equation (33) shows that pq − qp is not of the Hermite type, since it is a matrix with an imaginary diagonal.

The equations which the different writers finally obtain are then as follows:

$$pq - qp = \frac{h}{2\pi i}\mathbf{1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$H = \text{a diagonal matrix} = \{H_{nn}\}$$

$$h\nu_{nm} = H_{nn} - H_{mm}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where h is Planck's constant. Dirac introduced the following symbol, which he calls a " Poisson bracket ":

$$[x, y] = \frac{2\pi i}{h}(y \cdot x - x \cdot y), \quad . \quad . \quad . \quad . \quad (35)$$

so that equation (33) takes the form

$$[q, p] = \mathbf{1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33a)$$

The following properties exhibit the analogy between this operation and the Poisson brackets of classical mechanics:

$$\left.\begin{array}{ll} (a) & [x_1 + x_2, y] = [x_1, y] + [x_2, y] \\ (b) & [x_1x_2, y] = x_1[x_2, y] + [x_1, y]x_2 \\ (c) & [x, y] = -[y, x] \\ (d) & [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \end{array}\right\} . \quad . \quad (36)$$

The following are easily deduced from (33a):

$$\left.\begin{array}{l} [q, p^2] = p[q, p] + [q, p]p = 2p \\ [q^2, p] = q[q, p] + [q, p]q = 2q \end{array}\right\}, \quad . \quad . \quad (37)$$

and, by repeating the process,

$$\left.\begin{array}{l} [q, p^n] = np^{n-1} = \dfrac{\partial p^n}{\partial p} \\ [q^n, p] = nq^{n-1} = \dfrac{\partial q^n}{\partial q} \end{array}\right\} . \quad . \quad . \quad . \quad . \quad (38)$$

Thus if we take a function $f(q, p)$ which does not contain any other quantity which cannot be commuted with q or p, we can write

$$[f, p] = \frac{\partial f}{\partial q}; \quad [q, f] = \frac{\partial f}{\partial p}. \quad . \quad . \quad . \quad . \quad (39)$$

The partial derivatives are supposed to be defined by formula (27) of the last section, and are calculated by means of the ordinary rules.

In order to prove the formulæ (39), it suffices to assume that

f is expanded in a series of powers of p and q and to apply formula (38) to each term of the series.

All the above relations are still valid if p and q, not being matrices, correspond to symbols which obey our fundamental laws, just as we remarked at the end of the last section.

5. **The Search for a Solution; the Formal Analogy with Classical Mechanics.**—Returning to the case of a motion with one degree of freedom, the solution of which presents itself in the form of a matrix, let us investigate how to set about finding a solution. Let p and q be any two matrices whatever which satisfy relation (33); if we substitute them in the expression for H, we shall not in general obtain a diagonal matrix, but we can try to obtain this result by working with a canonical substitution (§ 3, equation (28)),

$$q' = S \cdot q \cdot S^{-1}; \quad p' = S \cdot p \cdot S^{-1}; \quad H' = S \cdot H \cdot S^{-1}, \quad . \quad (40)$$

which gives a new pair of matrices q', p' satisfying the relation (33).

Now suppose that we have found matrices q, p which are of the Hermite type and which make H a diagonal matrix. We may attempt to obtain other solutions by working with suitably chosen substitutions (40). If we take a diagonal matrix S we have (by equation (15)):

$$S = \{S_{nn}\}; \quad S^{-1} = \left\{\frac{1}{S_{nn}}\right\}, \quad . \quad . \quad . \quad . \quad (41)$$

which gives $$q'_{nm} = \sum_k \sum_l S_{nk} q_{kl} \frac{1}{S_{lm}} = \frac{S_{nn}}{S_{mm}} q_{nm}.$$

This transformation does not alter the diagonal terms, or the matrix H: further, it must not affect the fact that the matrices p, q are of the Hermite type, the conditions for which are:

$$\frac{S_{nn}}{S_{mm}} = \left(\frac{S_{mm}}{S_{nn}}\right)^* \quad \text{or} \quad \frac{S_{nn}\, S^*_{nn}}{S_{mm}\, S^*_{mm}} = \left|\frac{S_{nn}}{S_{mm}}\right|^2 = 1. \quad . \quad . \quad (42)$$

The coefficients S_{nn} must therefore be of the form

$$S_{nn} = e^{i\alpha_n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42a)$$

and the transformation reduces to this, that each term q_{nm} (or p_{nm}) is multiplied by $e^{i(\alpha_n - \alpha_m)}$. In particular, we may take

$$S_{nn} = e^{\frac{2\pi i}{h} H_{nn} t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

We see, in virtue of the quantum relations (34), that this operation results in the multiplication of each term q_{nm} or p_{nm} by $e^{\frac{2\pi i}{h}(H_{nn} - H_{mm})t}$, so that our matrices take the form (30).

I have dwelt to some extent on these transformations in order

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to show clearly the independence of the successive operations and the absence of contradiction between the conditions (33) and (34). The investigation of the coefficients q_{nm} and p_{nm} is quite independent of the determination of the frequencies; as for the coefficients, they themselves are determinate, apart from a phase factor $e^{i(\alpha_n - \alpha_m)}$, but the intensities

$$|q_{nm}|^2 = q_{nm}\, q^*_{nm} = q_{nm}\, q_{mn}$$

have an absolute meaning; they will be the intensities of the spectral lines ν_{nm}, which we shall thus obtain directly, without using the principle of correspondence or any special hypotheses.

From the solution obtained we can deduce an infinite number of others by simultaneous permutation of rows and columns, since this is an operation of canonical type (§ 2, equation (18*a*)) and it leaves the diagonal form of H intact; we can use this permutation to arrange the terms H_n so that they depend on n in a way which is continuous and as simple as possible.

The general equations (33) and (34) lead to a very simple result relating to the time-derivative $\dot{\mathrm{g}}$ of any matrix g. If the latter is given by

$$\mathrm{g} = \{g_{nm}\, e^{i2\pi\nu_{nm}t}\},$$

its derivative may be written

$$\dot{\mathrm{g}} = \{2\pi i \nu_{nm} g_{nm}\, e^{i2\pi\nu_{nm}t}\} = \frac{2\pi i}{h}\{(H_n g_{nm} - g_{nm} H_m)\, e^{i2\pi\nu_{nm}t}\} = [\mathrm{g}, \mathrm{H}].$$

To say that H is a diagonal matrix, or that the formula

$$\dot{\mathrm{g}} = [\mathrm{g}, \mathrm{H}] \qquad . \; . \; . \; . \; . \; . \; . \quad (44)$$

holds, are two equivalent statements; but the second will apply to quantities g, H, which obey the general rules of § 2, without, however, being capable of representation in matrix form. Thus the most general statement of the laws of quantum mechanics will be given by equations (33*a*) and (44).

The formulæ which we have obtained may be exhibited in a form which shows the analogy with classical mechanics. By applying (44) and (39) we have

$$\dot{\mathrm{q}} = [\mathrm{q}, \mathrm{H}] = \frac{\partial \mathrm{H}}{\partial \mathrm{p}}; \quad \dot{\mathrm{p}} = [\mathrm{p}, \mathrm{H}] = -\frac{\partial \mathrm{H}}{\partial \mathrm{q}}. \quad . \; . \quad (45)$$

If we make h tend to zero, formula (33) shows that the variables p and q will, in the limit, become interchangeable; the function H will reduce to the classical form, and we shall obtain the usual system of equations of motion. The matrix H is a diagonal matrix,

that is, it is independent of the time, and the energy remains constant.

As Dirac showed in his first paper, the Poisson brackets correspond to the similarly-named expressions in rational mechanics.

Let us give an example of calculations carried out by means of the fundamental formulæ. Suppose we wish to find the value of $\frac{d}{dt}(x^{-1})$; we have

$$\frac{d}{dt}(x^{-1} \cdot x) = \frac{d}{dt}(1) = [1 \cdot H] = 0,$$

whence, by applying (24),

$$0 = \frac{d}{dt}(x^{-1} \cdot x) = \frac{dx^{-1}}{dt} \cdot x + x^{-1} \cdot \dot{x};$$

multiply by x^{-1} on the right and we obtain

$$\frac{d}{dt}x^{-1} = -x^{-1} \cdot \dot{x} \cdot x^{-1}. \quad . \quad . \quad (46)$$

6. **The Harmonic Vibrator.**—Before continuing the general treatment, we shall deal with this particular case, which will illustrate the preceding remarks. The energy H will be assumed to be of the ordinary form

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 q^2; \quad \omega_0 = 2\pi\nu_0. \quad . \quad . \quad . \quad (47)$$

Formulæ (45) then give

$$\dot{q} = [q, H] = \frac{\partial H}{\partial p} = p,$$

and similarly

$$\dot{p} = -\omega_0^2 q,$$

so that we obtain the classical equation of motion,

$$\ddot{q} + \omega_0^2 q = 0. \quad . \quad . \quad . \quad . \quad . \quad (47a)$$

We see at once that the frequencies must be

$$\nu^2 = \nu_0^2; \qquad \nu = \pm \nu_0.$$

The energy terms H_n thus form a series of numbers differing among themselves by $h\nu_0$; in accordance with the observation made in the last section, we can arrange them in logical order, and write

$$H_n = (n + \alpha)h\nu_0 \qquad 0 < \alpha < 1. \quad . \quad . \quad . \quad (48)$$

Since the coefficients q_{nm}, p_{nm} are imaginary, *a priori* there is nothing to prevent the energy from having negative values; n may therefore be any integer, positive or negative.

The frequencies $\nu_{nm} = (n - m)\nu_0$ appear only for $n = m \pm 1$ (i.e. $\nu = \pm \nu_0$); thus the only coefficients q_{nm} or p_{nm} which do not vanish will be those for which the indices differ only by unity. The calculations are then simplified; from the matrix pq − qp pick out the n, n term, which, by the relation (33), must be equal to $\frac{h}{2\pi i}$:

$$(pq - qp)_{n,n} = \Sigma(\dot{q}_{nk} q_{kn} - q_{nk} \dot{q}_{kn}) = \dot{q}_{n,n+1} q_{n+1,n} - q_{n,n+1} \dot{q}_{n+1,n} + \dot{q}_{n,n-1} q_{n-1,n} - q_{n,n-1} \dot{q}_{n-1,n}$$

$$= 4\pi i \nu_0 (|\, q_{n-1,n} \,|^2 - |\, q_{n,n+1} \,|^2) = \frac{h}{2\pi i},$$

provided
$$|\, q_{n,n+1} \,|^2 - |\, q_{n-1,n} \,|^2 = \frac{h}{8\pi^2 \nu_0}. \quad . \quad . \quad . \quad (49)$$

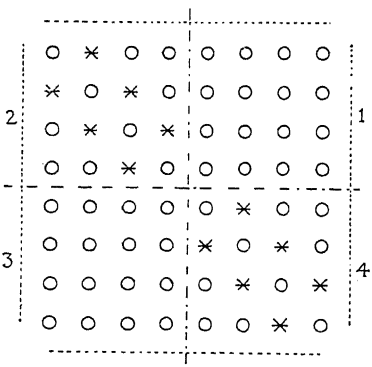
Calculation of the term H_n, with similar transformations, gives

$$|\, q_{n,n+1} \,|^2 + |\, q_{n-1,n} \,|^2 = \frac{H_n}{4\pi^2 \nu_0^2} = \frac{h}{4\pi^2 \nu_0}(n + \alpha). \quad . \quad (50)$$

This double system of equations has the solution

$$|\, q_{n,n+1} \,|^2 = \frac{h}{8\pi^2 \nu_0}(n + \alpha + \tfrac{1}{2}). \quad . \quad . \quad . \quad (51)$$

Thus it seems impossible to specify the value of the constant α: it must be noted that our solution involves matrices which are infinite in every sense, the indices n and m being capable of assuming any positive or negative integral values whatever. Born confined himself to the case of matrices with positive indices, but there is *a priori* no reason for this restriction.



For $\alpha = 0$ we obtain a level of zero energy $H_0 = 0$.

If we put $\alpha = \frac{1}{2}$, we have two zero constituents in the matrix q (or p):

$$q_{-1,0} = q_{0,-1} = 0.$$

This matrix then takes the form opposite, where the crosses indicate non-zero terms; we can cut the matrix twice, along the broken lines. All the operations of multiplication, addition, &c., will be carried out separately for the terms included in the second and fourth quadrants. The matrix can be decomposed into two

matrices, one with its indices all positive and the other with its indices all negative.

Born and Heisenberg found for their solution * only the matrix with positive indices which results in this way from the case $\alpha = \frac{1}{2}$. It gives a series of levels of energy which are all positive, the lowest being $\frac{1}{2}h\nu_0$:

$$H_n = (n + \tfrac{1}{2})h\nu_0.$$

The second matrix, with all its indices negative, is, however, equally a solution, and it gives negative levels of energy, $-(n + \frac{1}{2})h\nu_0$; and the general solution (48) is perfectly acceptable.

Thus it appears from this example that several solutions for a single problem may exist, the choice between which is arbitrary. Moreover, the limitation of the matrices is not obligatory, and some problems † can only be solved by matrices unlimited in every direction, without its being possible to find cases of decomposition like the above.

7. **Classical Mechanics as the Limit for large Quantum Numbers.**—The formal analogies which we indicated at the end of § 5 are very interesting, but they do not exempt us from proceeding to a further verification; we shall investigate what the quantum laws become when they are applied to the terms q_{nm}, p_{nm} of matrices for which the indices n and m are very large; ‡ in this way we shall verify that our statements satisfy Bohr's principle of correspondence and yield the rules of classical mechanics, when states with large quantum numbers are considered.

The numbering of the terms has no absolute meaning in reality, since simultaneous permutation of the rows and columns may always be carried out; we therefore assume that the terms have been arranged by means of a permutation of this kind in such an order that they satisfy the following conditions, when the indices n and m are very large:

I. The coefficients a_{nm} of a matrix a are continuous functions of n which can be expanded in powers of n by Taylor's series, for every value of $m - n = r$ (i.e. for every step r starting from the diagonal).

II. The absolute values of a_{nm} tend to 0 when r becomes very large.

With these assumptions, we may write

$$H_m = H_n + (m - n)\frac{\partial H_n}{\partial n}, \quad . \quad . \quad . \quad . \quad (52)$$

* B. I, p. 875. † L. Brillouin, *C. R.*, Vol. **182** (1926), p. 374.

‡ Here we give in precise terms an account of the considerations by means of which Heisenberg and later Dirac obtained the enunciations applicable to matrices in their first papers.

expanding the terms of the diagonal matrix H_n in series. From this, we obtain for the frequencies the values

$$\nu_{mn} = \frac{1}{h}(H_m - H_n) = (m-n)\frac{1}{h}\frac{\partial H_n}{\partial n} = r\nu_n, \quad . \quad (53)$$

where

$$\nu_n = \frac{1}{h}\frac{\partial H_n}{\partial n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Thus all the frequencies are harmonics of the fundamental frequency ν_n, which slowly varies with the number n. The harmonics of high order correspond to terms whose distance r from the diagonal is large.

In this way we see that our second assumption reduces to this, that the amplitudes of the harmonics of high order are supposed negligible; this condition is satisfied in every case where the trajectory does not exhibit sharp angles or discontinuities in the velocities.

In virtue of the first assumption, we shall put

$$\left.\begin{aligned} p_{n,\,n+r} &= p_{n_0,\,n_0+r} + (n-n_0)\varpi_{n_0,\,n_0+r} + \ldots \\ q_{n,\,n+r} &= q_{n_0,\,n_0+r} + (n-n_0)\chi_{n_0,\,n_0+r} + \ldots \end{aligned}\right\} \quad . \quad (55)$$

and in every case where differences of neighbouring terms are not involved we can replace $q_{n,\,n+r}$ by $q_{n+s,\,n+s+r}$, where s will not be too great.

It is to be noted that none of the components $q_{n,\,n+r}$ are comparable to the co-ordinates of classical mechanics; $q_{n,\,n+r}$ is merely the amplitude of a harmonic. An atom in the state n (that of energy H_n) can radiate all the frequencies which correspond to transitions towards states of less energy; the series of terms $q_{n,\,n+r}$ for which H_{n+r} is less than H_n represents the actually observed motion. In order to have a real expression, we shall therefore take

$$q_n = \Sigma(q_{n,\,n-r}e^{i2\pi r\nu_n t} + q_{n-r,\,n}e^{-i2\pi r\nu_n t}), \quad . \quad . \quad (56)$$

where the values of r range between 0 and $+\infty$ or else between $-n$ and 0, according to the way in which H_n varies as a function of n. This is the expansion we wrote down previously (§ 1, formula (5)).

We can simplify it here, for $q_{n-r,\,n}$ is nearly equal to $q_{n,\,n+r}$ (assumption I), which in both of the above cases gives

$$q_n = \sum_{r=-\infty}^{r=+\infty}(q_{n,\,n-r}e^{i2\pi r\nu_n t}). \quad . \quad . \quad . \quad . \quad . \quad (57)$$

In the expansion of the momentum p_n, we replace $p_{n,\,n-r'}$ by $p_{n+r',\,n}$:

$$p_n = \sum_{r'=-\infty}^{r'=+\infty}(p_{n+r',\,n}e^{i2\pi r'\nu_n t}); \quad . \quad . \quad . \quad . \quad (58)$$

we can extend the summations from $r = -\infty$ to $r = +\infty$, since n is very large and the harmonic terms of high order are negligible.

Let us then form the integral of action, taken over a complete oscillation:

$$\mathcal{J}_n = \int p_n \, dq_n = \int p_n \dot{q}_n \, dt. \quad . \quad . \quad . \quad . \quad (59)$$

We have $$\dot{q}_n = 2\pi i \nu_n \sum_r r q_{n,\,n-r} \, e^{i 2\pi r \nu_n t};$$

$$p_n \dot{q}_n = 2\pi i \nu_n \sum_s e^{i 2\pi s \nu_n t} \sum_r r q_{n,\,n-r} \, p_{n+s-r,\,n}$$

by grouping together the terms which give the same harmonic $s = r + r'$.

When we integrate over a complete period $\frac{1}{\nu_n}$, all the terms vanish, except those for which $s = 0$.

$$\mathcal{J}_n = 2\pi i \sum r q_{n,\,n-r} \, p_{n-r,\,n}. \quad . \quad . \quad . \quad . \quad (60)$$

I wish to find how $\mathcal{J}_n$ varies as a function of the number n, the latter being assumed very large; for a variation $\Delta n = 1$, I obtain:

$$\frac{\partial \mathcal{J}_n}{\partial n} = [\mathcal{J}_n]_n^{n+1} = 2\pi i \sum_r r [q_{n,\,n-r} \, p_{n-r,\,n}]_n^{n+1} = 2\pi i \sum_r [q_{n,\,n-r} \, p_{n-r,\,n}]_n^{n+r}.$$

The brackets indicate the differences of the values taken for $n + 1$ and n or for $n + r$ and n; the latter difference is clearly (assumption I) r times as great as the former.

Thus we obtain

$$\frac{\partial \mathcal{J}_n}{\partial n} = 2\pi i \sum_r (q_{n+r,\,r} \, p_{n,\,n+r} - q_{n,\,n-r} \, p_{n-r,\,n})$$

$$= 2\pi i (pq - qp)_{n,\,n} = n, \quad . \quad . \quad . \quad (61)$$

so that for large numbers we find that

$$\mathcal{J}_n = nh + \alpha = \text{constant}. \quad . \quad . \quad . \quad . \quad (62)$$

This equation (62) is identical with that given by classical mechanics to determine the motion: moreover, the classical frequency is given by the relation

$$\nu_n = \frac{\partial H}{\partial \mathcal{J}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

which coincides with that found in (54). The agreement is therefore complete, and the matrix mechanics gives in the limit, for large quantum numbers, a discontinuous set of motions, all of which satisfy the laws of classical mechanics.

Of the relations (33), we used in (61) only those which appear on

the diagonal, that is to say $(pq - qp)_{n,n} = \frac{h}{2\pi i}$; it is of some interest to find what the other conditions mean:

$$(pq - qp)_{nm} = \sum_k (p_{nk} q_{km} - q_{nk} p_{km}) = 0 \qquad n \neq m. \quad (64)$$

Let us take the integral of action for any length of time t, differing from a complete period,

$$\int_0^t p_n \dot{q}_n \, dt = \mathcal{J}_n t \nu_n + \sum_s \frac{1}{s} (e^{i2\pi s \nu_n t} - 1) \sum r q_{n, n-r} p_{n-r, n-s}. \quad (65)$$

We have replaced $p_{n+s-r,n}$ by its equivalent $p_{n-r,n-s}$, and we note that in the summation with respect to s the value $s = 0$ is excluded, the terms corresponding to the case $s = 0$ having been integrated in the first term $\mathcal{J}_n t \nu_n$. We know that in classical mechanics the integral (65) does not vary if we pass from the actual orbit to a path infinitely close to it without changing $\mathcal{J}_n$ or t; we may assume that the variation of p_n and q_n is equivalent to a variation of unity, for example, in the index n; in order to have

$$\delta \int_0^t p_n \, dq_n = 0$$

we shall set

$$0 = \sum_r r \left| q_{n, n-r} p_{n-r, n-s} \right|_n^{n+1} = \sum_r \left| q_{n, n-} p_{n-r, n-s} \right|_n^{n+r}$$

$$= \sum (q_{n+r, n} p_{n, n+r-s} - q_{n, n-r} p_{n-r, n-s}),$$

which reduces to the condition (64): we have only to put $n + r - s = k$ and $n - s = m$, and replace $q_{n+r,n}$ by its equivalent $q_{n+r-s,n-s} = q_{k,m}$, in the first term; in the second product we have $n - r = k$ and $n - s = m$.

8. **Systems with several Degrees of Freedom: Quasi-periodic Cases.**—The equations written down for a single co-ordinate may be generalized for any number of co-ordinates without difficulty, and the solution presents itself in the form of matrices in the case of a problem of quasi-periodic (or multi-periodic)* type. It must, however, be assumed that *no constraint* exists, so that the number of degrees of freedom is equal to f, the number of co-ordinates introduced.† Each co-ordinate q_j and each momentum p_j will be represented by a matrix of $2f$ dimensions, that is to say, by an array of terms like $q_j(n_1, n_2, \ldots, n_f; m_1, m_2, \ldots, m_f)$ depending on two

* [M. Born, *loc. cit.*, Chapter II; G. Birtwistle, *loc. cit.*, Chapters IV, VII.]

† The case of a rigid rotating system, for example, is not of this type, and cannot be solved in the way indicated here. See L. Brillouin, *C. R.*, Vol. **182** (1926), p. 374.

series of whole numbers n and m; the rules of multiplication (10) may then be written

$$(pq)(n_1,\ldots,n_f;\ m_1,\ldots,m_f) = \sum_{k_1}\ldots\sum_{k_f} p(n_1,\ldots,n_f;\ k_1,\ldots,k_f)\,q(k_1,\ldots,k_f;\ m_1,\ldots,m_f). \quad (66)$$

The unit matrix **1** has for its components $\delta_{n_1m_1}, \delta_{n_2m_2}, \ldots, \delta_{n_fm_f}$, where the coefficients δ_{rs} are equal to unity if $r = s$ and vanish if $r \neq s$. The q's (and the p's) are matrices of the Hermite type, and we have

$$q_j(n_1,\ldots,n_f;\ m_1,\ldots,m_f) = q^*{}_j(m_1,\ldots,m_f;\ n_1,\ldots,n_f).$$

The fundamental operations are then carried out exactly as was indicated in the second section, and all our definitions are valid. The fundamental equations of quantum mechanics are as follows:

$$[q_r, q_s] = 0;\quad [p_r, p_s] = 0;\quad [q_r, p_s] = \delta_{rs} = \begin{cases} 1 \text{ if } r = s \\ 0 \text{ if } r \neq s. \end{cases} \quad (67)$$

The co-ordinates q are interchangeable among themselves, and so are the momenta p: multiplication ceases to be commutative only when it is carried out with a co-ordinate q_r and the corresponding momentum p_r.

A particular problem is defined by the expression for its Hamiltonian function $H(q_1,\ldots,q_f;\ p_1,\ldots,p_f)$, for which the symmetrized form which follows from the classical function is taken. In order to solve the problem, we must make H a diagonal matrix, that is, it must be reduced to the terms $n_1 = m_1$, $n_2 = m_2$; $\ldots n_f = m_f$; the frequencies are then given by the rule

$$h\nu(n_1,\ldots,n_f;\ m_1,\ldots,m_f) = H(n_1,\ldots,n_f) - H(m_1,\ldots,m_f). \quad (68)$$

The formulæ indicated in § 4 all continue to apply: in particular, the relations (39) give

$$[f, p_k] = \frac{\partial f}{\partial q_k} \text{ and } [q_k, f] = \frac{\partial f}{\partial p_k}. \quad . \quad . \quad . \quad (69)$$

If the function f depends on the q's and p's, it in fact does not contain any quantity which is not commutable with q_k, except p_k. Formula (44) is still valid:

$$\dot{g} = [g, H], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

so that our equations again reduce to the canonical form:

$$\dot{q}_k = (q_k, H) = \frac{\partial H}{\partial p_k};\quad \dot{p}_k = [p_k, H] = -\frac{\partial H}{\partial q_k}. \quad . \quad (70)$$

The equations are invariant with respect to a canonical trans-

formation $S \cdot q \cdot S^{-1}$, and also with respect to a linear orthogonal transformation of the q's and p's; the conditions

$$q'_k = \sum_l a_{kl} q_l; \quad p'_k = \sum_l a_{kl} p_l, \text{ with } \sum_h a_{kh} a_{lh} = \delta_{kl}, \qquad (71)$$

which represent this transformation, correspond to a rotation of rectangular co-ordinates: it is at once obvious that

$$[q'_l, p'_k] = \sum_h \sum_j a_{kh} a_{lj} [q_j, p_h] = \delta_{kl}.$$

9. **The Investigations of a Perturbed System.**—This type of problem frequently turns up in atomic mechanics.* The influence of a constant electric or magnetic field leads to the explanation of the Stark and Zeeman effects, and at the same time gives information about the dielectric or magnetic properties of matter. Suppose then that an atomic system is disturbed by an external cause, and let the expression for the energy be

$$H(p, q) = H_0(p, q) + \lambda H_1(p, q) + \lambda^2 H_2 + \ldots, \quad . \quad (72)$$

where λ represents a very small constant parameter. Let $p_0(n, m)$ and $q_0(n, m)$ be the components of the matrices p_0, q_0 which give the solution of the problem without perturbations ($\lambda = 0$). These matrices satisfy the fundamental equations, and make H_0 a diagonal matrix. Let $W_{0,m}$ be the components of this latter matrix: the frequencies of the undisturbed system are

$$\nu_0(n, m) = \frac{1}{h}(W_{0,n} - W_{0,m}). \quad . \quad . \quad . \quad . \quad (73)$$

In accordance with the general observations in § 5, we shall, in order to solve the problem (72), first try to find a set of coefficients $p(n, m)$, $q(n, m)$ which satisfy the conditions (33) and make H a diagonal matrix; the frequencies will afterwards be determined by the quantum rule.

To find the quantities $p(n, m)$, $q(n, m)$, we shall start from the quantities $p_0(n, m)$, $q_0(n, m)$ and try to find a canonical transformation

$$p = S p_0 S^{-1}; \quad q = S q_0 S^{-1} \quad . \quad . \quad . \quad . \quad (74)$$

which makes H a diagonal matrix. Let us put

$$\left.\begin{aligned} S &= 1 + \lambda S_1 + \lambda^2 S_2 + \ldots \\ S^{-1} &= 1 - \lambda S_1 + \lambda^2 (S_1^2 - S_2) + \ldots \end{aligned}\right\} \quad . \quad . \quad (75)$$

By equation (29), the matrix H is expressed by:

* [M. Born, *loc. cit.*, Chapter IV; G. Birtwistle, *loc. cit.*, Chapters XX, XXI.]

$$\left.\begin{aligned} H(p, q) &= S \cdot H(p_0, q_0) S^{-1} \\ &= H_0(p_0 q_0) \\ &+ \lambda[S_1 \cdot H_0(p_0, q_0) - H_0(p_0, q_0) \cdot S_1 + H_1(p_0, q_0)] \\ &+ \lambda^2[S_2 H_0 - H_0 S_2 + H_0 S_1^2 - S_1 H_0 S_1 + S_1 H_1 \\ &\qquad - H_1 S_1 + H_2] + \dots \\ &+ \lambda^r[S_r H_0 - H_0 S_r + F_r(H_0, H_1 \dots H_r, S_1 \dots S_{r-1})] + \dots \end{aligned}\right\} \quad (76)$$

We shall write down the condition that at each stage of approximation the terms obtained reduce to the diagonal, and for the matrix H we shall obtain an expansion in powers of the parameter λ:

$$H_m = W_{0,m} + \lambda W_{1,m} + \dots + \lambda^r W_{r,m} + \dots . \quad (77)$$

Take, for example, the terms in λ^r: we must write

$$S_r H_0 - H_0 S_r + F_r = W_r, \quad . \quad . \quad . \quad . \quad (78)$$

where W_r is a diagonal matrix. The expression F_r is known, since the quantities $S_1, \dots, S_{r-1}$ have been determined by preceding approximations; moreover, the first two terms reduce (by formulæ (44) and (73)) to

$$(S_r H_0 - H_0 S_r)_{n,m} = - h\nu_0(n, m) S_r(n, m), \quad . \quad (78a)$$

where $\nu_0(n, m)$ is a frequency of the undisturbed motion. This expression vanishes on the diagonal, since $\nu_0(m, m)$ is zero by (34), so that we are left with the relation

$$W_{r,m} = F_r(m, m), \text{ or } W_r = \overline{F}_r, \quad . \quad . \quad . \quad (79)$$

for the determination of the $W_{r,m}$'s.

The matrix S_r is then given by

$$S_r(n, m) = \frac{1}{h\nu_0(n, m)} F_r(n, m) \qquad n \neq m. \quad . \quad . \quad (80)$$

The diagonal terms $S_r(m, m)$ are indeterminate and may be taken as zero. Thus we can form the matrix S step by step, and then find the p's and q's which give the solution of the problem, by applying the formulæ (74). Thereafter the frequencies are given by the conditions

$$\nu(n, m) = \frac{1}{h}(H_n - H_m) = \nu_0(n, m) + \frac{\lambda}{h}(W_{1,n} - W_{1,m}) + \dots . \quad (81)$$

We have yet to verify that the matrices found are of the Hermite type, i.e. that they satisfy the relation

$$q^* = \tilde{q},$$

where the sign $\sim$ indicates a matrix in which rows and columns have been interchanged. Now, we can verify that the solution S is such that

$$S \cdot \tilde{S}^* = 1,$$

which involves

$$q^* = S^* q_0^* S^{-1*} = \tilde{S}^{-1} \cdot \tilde{q}_0 \cdot \tilde{S} = \tilde{q}, \quad . \quad . \quad . \quad (82)$$

bearing the obvious relation $(\widetilde{ab}) = \tilde{b}\,\tilde{a}$ in mind.

These general formulæ are of great importance, and they illustrate the flexibility of the methods of calculation of this mechanics of the atom. As an example, we shall write down the results of the first approximation in detail; we obtain

$$W_1 = \bar{H}_1 \quad \text{and} \quad S_1(n, m) = \frac{H_1(n, m)}{h\nu_0(n, m)}. \qquad n \neq m. \quad . \quad (83)$$

In this formula, as in (79), the bar above the letter stands for the mean value of a matrix, that is, the matrix reduced to its diagonal. Substituting the above values (83) in formulæ (74) and (75), we deduce that

$$q = q_0 + \lambda q_1 + \ldots, \quad \text{with} \quad q_1 = S_1 q_0 - q_0 S_1,$$

so that, by (83),

$$q_1(n, m) = \frac{1}{h} \Sigma' \left[\frac{H_1(n, k) q_0(k, m)}{\nu_0(n, k)} - \frac{q_0(n, k) H_1(k, m)}{\nu_0(k, m)} \right], \quad (84)$$

with similar formulæ for p: the accent attached to the sign Σ is to remind us that the summation is to be carried out for all terms where the denominator $\nu_0(n, k)$ or $\nu_0(k, m)$ does not vanish.

It is very remarkable to find that the formulæ (84) which we have just found are identical with those obtained by Kramers, in his theory of dispersion, for the limiting case of very low frequencies. To make the comparison we have only to put $\lambda H_1 = qeE$, and call e the electric charge of the particle and E the applied field.

We have stated the formulæ for one degree of freedom, but they apply equally well to cases with any number of co-ordinates whatever; we have merely to suppose that n and m stand for the series of numbers $n_1, \ldots, n_f$ and $m_1, \ldots, m_f$.

10. **Coupling of Systems Originally Separate.**—This example will illustrate the application of the previous method to several degrees of freedom. Suppose that the function H takes the following form:

$$H = H_{0(1)}(p_1, q_1) + H_{0(2)}(p_2, q_2) + \ldots + H_{0(f)}(p_f, q_f) + \lambda H_1(p_1, \ldots, p_f;\ q_1, \ldots, q_f) + \lambda^2 H_2 + \ldots. \quad (85)$$

For $\lambda = 0$ we have f separate systems, each with one degree of freedom; when λ is not zero, we couple these different oscillators. Let us first examine the case $\lambda = 0$.

Assume that each problem has been solved, by means of matrices with two coefficients,

$$\mathrm{p}_k = \{p_k(n_k, m_k)\}; \quad \mathrm{q}_k = \{q_k(n_k, m_k)\}; \quad . \quad . \quad (86)$$

the quantities p_k, q_k make $\mathrm{H}_{0(k)}$ a diagonal matrix

$$\mathrm{H}_{0(k)} = \{H_{0(k)}(n_k, n_k)\} \quad . \quad . \quad . \quad . \quad (87)$$

and satisfy the equations

$$[\mathrm{q}_k, \mathrm{p}_k] = \mathbf{1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

The solution will take the form of a matrix of $2f$ dimensions if we set

$$\mathrm{q}_{0(k)} = \{\delta_1 \delta_2 \ldots \delta_{k-1} \delta_k \ldots \delta_f q_k(n_k, m_k)\}, \quad . \quad (89)$$

where for short we have put δ_f for $\delta_{n_f m_f}$: the quantities δ_f are equal to unity when $n_f = m_f$ and vanish in other cases. For $\mathrm{p}_{0(k)}$ we assume an expression of the same type as (89), and we see that our p's and q's satisfy all the equations (67). For H we obtain a diagonal matrix

$$H_0(n_1 \ldots n_f) = H_{0(1)}(n_1)\delta_2\delta_3 \ldots \delta_f + \ldots$$
$$+ \delta_1 \ldots \delta_{k-1}\delta_{k+1} \ldots \delta_f H_{0(k)}(n_k) + \ldots + \delta_1 \ldots \delta_{f-1} H_{0(f)}(n_f). \quad (90)$$

The initial separate motions gave frequencies

$$\nu_{0(k)}(n_k, m_k) = \frac{1}{h}[H_{0(k)}(n_k) - H_{0(k)}(m_k)]: \quad . \quad . \quad (91)$$

at first sight it seems, from (90), that we could now find all the combination-frequencies

$$\nu_0(n_1, \ldots, n_f; m_1, \ldots, m_f) = \sum_k \nu_{0(k)}(n_k, m_k). \quad . \quad (92)$$

A restriction immediately appears, however; only those frequencies are observable for which there is a non-vanishing amplitude term $q_k(n_1, \ldots, n_f; m_1, \ldots, m_f)$. Let us therefore go back to the formulæ (89); they show that the coefficient $q_k(n_1, \ldots, n_f; m_1, \ldots, m_f)$ differs from zero only if $n_1 = m_1$, $n_2 = m_2$, . . ., $n_{k-1} = m_{k-1}$; $n_{k+1} = m_{k+1}$, . . ., $n_f = m_f$, the numbers n_k and m_k alone being unequal. This means that we can observe only the frequencies $\nu_{0(k)}$ of formula (91): the systems exist side by side, without any modification. I have dwelt on this example to some extent, in order to show clearly how the matrix mechanics automatically yields a selection of the observable

frequencies, whereas the old quantum theory required to introduce the principle of correspondence for this.

The combination-frequencies (92) will only appear when we couple the different systems ($\lambda \neq 0$); and in different cases we shall find a greater or lesser number appearing, with or without displacements. We shall then start from the quantities $p_{0(k)}$, $q_{0(k)}$ written down in (89), and we shall try to find a canonical transformation with a function

$$S = 1 + \lambda S_1 + \lambda^2 S_2 + \dots$$

which makes expression (85) a diagonal matrix: the co-ordinates will then be

$$q_k = S \cdot q_{0(k)} \cdot S^{-1}$$

in accordance with the general rule.

11. **Kramers' Dispersion Formulæ.***—Consider an atomic system, which, for greater simplicity of writing, we shall assume to have only one degree of freedom; let $q_{0(1)}$ be a co-ordinate and $p_{0(1)}$ the corresponding momentum, and let $H_{0(1)}$ be the energy function. We wish to investigate the perturbation of this system by an external force which is a sinusoidal function of the time with frequency ν, e.g. the electric field of an incident wave. We can reduce this problem to that of the perturbations dealt with in the last two sections.

Let $q_{0(2)}$ and $p_{0(2)}$ be the co-ordinate and momentum of a harmonic oscillator external to the atom; we have seen (§ 6) that the matrix $q_{0(2)}$ reduces to the terms

$$q_{0(2)}(n_2, n_2 + 1) = q_{0(2)}(n_2 + 1, n_2) = \sqrt{\frac{h}{8\pi^2 \nu}(n_2 + 1)}\, e^{\pm i 2\pi \nu t}, \quad (93)$$

while the levels of energy are †

$$H_{0(2)}(n_2) = (n_2 + \tfrac{1}{2})\, h\nu; \quad \dots \quad (93a)$$

from among the different solutions considered in § 6, we have chosen the simplest, that corresponding to $\alpha = \frac{1}{2}$.

We shall now consider the effect of coupling this oscillator and our atom, by assuming an energy function

$$H = H_{0(1)}(p_1, q_1) + H_{0(2)}(p_2, q_2) + \lambda H_1(p_1, q_1, q_2) + \dots . \quad (94)$$

We shall investigate a function of transformation

$$S = 1 + \lambda S_1 + \dots; \quad S^{-1} = 1 - \lambda S_1 + \dots,$$

* [See references in introductory paragraph.]

† Born (B. II, p. 570, equation (36)) takes a different form for the matrix $H_{0(2)}$, with which I have been unable to obtain the results announced by him; I have therefore modified this part of the exposition.

which will enable us to reduce H to the diagonal form with respect to the two series of whole numbers n_1, m_1 and n_2, m_2:

$$S \cdot H \cdot S^{-1} = \{ W_{0(1)}(n_1) + W_{0(2)}(n_2) + \lambda W_1(n_1, n_2) + \ldots \}. \quad (95)$$

We shall then have the following conditions:

$$W_{0(1)}(n_1) = H_{0(1)}(n_1); \quad W_{0(2)}(n_2) = H_{0(2)}(n_2) = (n_2 + \tfrac{1}{2})\, h\nu$$

and $$S_1 H_{0(1)} - H_{0(1)} S_1 + S_1 H_{0(2)} - H_{0(2)} S_1 + H_1 = W_1. \quad (96)$$

The first terms are easily rearranged to give

$$- S_1(n_1, n_2;\ m_1, m_2)\,[h\nu_{0(1)}(n_1, m_1) + h(n_2 - m_2)\,\nu] + H_1(n_1, n_2;\ m_1, m_2) = W_1(n_1, n_2),$$

whence we readily deduce, for the diagonal terms ($n_1 = m_1$, $n_2 = m_2$),

$$W_1(n_1, n_2) = H_1(n_1, n_2;\ n_1, n_2), \text{ that is, } W_1 = \overline{H}_1, \quad (97)$$

and $$S_1(n_1, n_2;\ m_1, m_2) = \frac{H_1(n_1, n_2;\ m_1, m_2)}{h[\nu_{0(1)}(n_1, m_1) + (n_2 - m_2)\nu]}$$

for $n_1 \neq m_1$ and $n_2 \neq m_2$. . (98)

To apply this to the particular problem we are dealing with, we must specify the form of the perturbation-function H_1; we shall, for example,* take

$$\lambda H_1 = \sqrt{\frac{2\pi^2\nu}{h}}\,(q_{0(2)}(0, 1) + q_{0(2)}(1, 0))\; q_{0(1)}\, eE; \quad . \quad (99)$$

in virtue of equations (93) this reduces to

$$\lambda H_1 = q_{0(1)}\, eE \cos 2\pi\nu t, \quad . \quad . \quad . \quad . \quad (99a)$$

where E represents the amplitude of the electric field and e the charge on the electron; we shall assume that the motion of the latter is represented by the terms of the matrix $q_{0(1)}$. The only values of n_2, m_2 which concern us are 0 and 1; we then have

$$\left.\begin{aligned}
\lambda H_1(n_1, 0;\ m_1, 1) &= \frac{Ee}{2}\, q_{0(1)}(n_1, m_1);\\
\lambda H_1(n_1, 1;\ m_1, 0) &= \frac{Ee}{2}\, q_{0(1)}(n_1, m_1);\\
\lambda S_1(n_1, 0;\ m_1, 1) &= \frac{Ee}{2h}\, \frac{q_{0(1)}(n_1, m_1)}{\nu_{0(1)}(n_1, m_1) - \nu};\\
\lambda S_1(n_1, 1;\ m_1, 0) &= \frac{Ee}{2h}\, \frac{q_{0(1)}(n_1, m_1)}{\nu_{0(1)}(n_1, m_1) + \nu}.
\end{aligned}\right\} \quad (100)$$

* Every combination $q_{0(1)} \underset{n, m}{\Sigma} A(n_2, m_2)\, q_{0(2)}(n_2, m_2)$ in which the sum Σ has a finite value $eE \cos 2\pi\nu t$ would lead to the same result.

Formula (97) then gives the mean energy of the atomic system disturbed by the oscillating field; if we revert to the co-ordinate $q_{1)}$ of the atom, we obtain

$$q_{1)} = S \cdot q_{0(1)} \cdot S^{-1} = q_{0(1)} + \lambda q_{1(1)} + \dots \quad . \quad (101)$$

with the following expansions:

$$\lambda q_{1(1)}(n_1, 0; m_1, 1) = \frac{Ee}{2h} \sum_k \left[\frac{q_{0(1)}(n_1, k)\, q_{0(1)}(k, m_1)}{\nu_0(n_1, k) - \nu} - \frac{q_{0(1)}(n_1, k)\, q_{0(1)}(k, m_1)}{\nu_0(k, m_1) - \nu} \right]. \quad (102)$$

The term in $q_{1(1)}$ $(n_1, 1; m_1, 0)$ is quite similar, with $+\nu$ in the denominators instead of $-\nu$.

The formulæ (101) and (102) represent Kramers' dispersion formula, which we mentioned in § 1. It gives the perturbations of the atomic system under the influence of an external alternating field, the latter being assumed weak. Let us make a clear statement as to the frequencies of the different terms: in (101), we first have terms of amplitude $\delta_{n_2 m_2} q_{0(1)}(n_1, m_1)$, to which the frequencies $\nu_{0(1)}$ (n_1, m_1) of the undisturbed motion correspond (the transition $n_1 - m_1$, with $n_2 = m_2$); as for the auxiliary terms $\lambda q_{1(1)}(n_1, 0; m_1, 1)$ and $\lambda q_{1(1)}(n_1, 1; m_1, 0)$, their frequencies are, as is easily verified, $\nu_{0(1)}(n_1, m_1) - \nu$ and $\nu_{0(1)}(n_1, m_1) + \nu$ respectively. Thus there are terms of the same frequency as the incident wave (in the case $n_1 = m_1$); this corresponds to classical diffusion and scattering; but all the combination-frequencies $\nu_0(n_1, m_1) \pm \nu$ are also foretold. Optical resonance corresponds to the case where the incident frequency ν approaches one of the absorption frequencies; this makes a series of denominators in the expansions (102) very small.

For very low frequencies we obtain formula (84) (§ 9) in the limit. For very high frequencies,* we can expand as follows:

$$\frac{1}{\nu_0(n_1, k) - \nu} = -\frac{1}{\nu}\left[1 + \frac{\nu_0(n_1, k)}{\nu} + \dots\right].$$

The terms in $\frac{1}{\nu}$ in the expansion (102) cancel, and we are left with

$$\lambda q_{1(1)}(n_1, 0; m_1, 1) = -\frac{Ee}{2h\nu^2} \sum_k [\nu_0(n_1, k)\, q_{0(1)}(n_1, k)\, q_{0(1)}(k, m_1) - \nu_0(k, m_1)\, q_{0(1)}(n_1, k)\, q_{0(1)}(k, m_1)]. \quad . \quad (103)$$

Suppose $q_{0(1)}$ is a Cartesian co-ordinate; we shall have $p_{0(1)} = M\dot{q}_{0(1)}$,

* The passage to the limit for very high frequencies, as indicated by Born (B. II, p. 572), does not seem to me to be correct; I have preferred to follow Kramers' discussion (*Physica*, Vol. **5** (1925), pp. 369–76).

where M is called the mass of the electron; the second member of (103) will then be

$$-\frac{Ee}{2h\nu^2}\frac{1}{2\pi iM}[p_{0(1)}q_{0(1)}-q_{0(1)}p_{0(1)}](n_1, m_1)$$
$$=\begin{cases}0 & \text{if } n_1 \neq m_1,\\ \dfrac{Ee}{8\pi^2M\nu^2} & \text{if } n_1 = m_1,\end{cases} \quad . \quad . \quad (104)$$

since the quantities $p_{0(1)}$, $q_{0(1)}$ satisfy the quantum conditions (33).

Thus there remain only the terms $n_1 = m_1$, the frequency of which is equal to that of the incident wave; by grouping together $q_{0(1)}(n_1, 0;\ n_1, 1)$ and $q_{0(1)}(n_1, 1;\ n_1, 0)$ we obtain:

$$\lambda q_{1(1)}(n_1, 0;\ n_1, 1)e^{-i2\pi\nu t}+\lambda q_{1(1)}(n_1, 1;\ n_1, 0)e^{i2\pi\nu t}$$
$$=\frac{Ee}{4\pi^2M\nu^2}\cos 2\pi\nu t. \quad . \quad . \quad . \quad . \quad (105)$$

The electron then behaves exactly like a free electron, and oscillates in phase with the field of the incident wave.

12. **Dirac's Generalizations.**—Wherever possible, we have exhibited the most general form of the equations we obtained; the fact that atomic quantities are capable of being represented by matrices does not seem essential. The set of formulæ is, however, equally applicable to any quantities whatever, provided they obey the fundamental rules (11), (13), and (14) of § 2.

Dirac therefore assumes that atomic quantities cannot be represented by ordinary co-ordinates or be expressed as functions of the cardinal numbers (c-numbers), but that they oblige us to introduce quantum numbers (q-numbers). Ordinary arithmetical operations may be carried out with these q-numbers, with the exception of the commutative law of multiplication. The definition of unity and of the inverse of a q-number are additional hypotheses which cannot be done without.* The mechanical laws are then expressed by the formulæ (67) and (70), provided the co-ordinates q_k are independent, without any condition of constraint.

How ought we to picture a quantum number to ourselves? It is impossible to do so in general, but if we are dealing with a quantity with multi-periodic variation, we can always return to the representation by matrices. The distinguishing feature of Dirac's methods of calculation is that, in order to solve a given problem, he does not hesitate to proceed by intermediate steps which cannot be repre-

* Dirac also introduces the condition that if the product a · b vanishes, one of the numbers a, b vanishes; we observed at the end of § 2 that this hypothesis is not true for matrices.

sented in matrices. Consider, for example, a problem in which the energy is of the form

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + F(\sqrt{x^2 + y^2}); \quad . \quad (106)$$

this corresponds to the motion of an electron in a central field of force, and the hydrogen atom is a particular case of this type. Dirac changes to polar co-ordinates by means of the formulæ

$$\left.\begin{aligned} &x = r\cos\Theta, \quad y = r\sin\Theta, \\ &p_r = \tfrac{1}{2}(p_x \cdot \cos\Theta + \cos\Theta \cdot p_x) + \tfrac{1}{2}(p_y \cdot \sin\Theta + \sin\Theta \cdot p_y) \\ &p_\theta = x \cdot p_y - y \cdot p_x. \end{aligned}\right\} \quad (107)$$

He verifies that p_r and p_θ really are the momenta conjugate to r and Θ, if x and y are two independent variables;* the equations (33*a*) apply to both systems of variables. The functions cos and sin are defined by the classical expansions in series. Referred to the new variables, the function H takes the following form:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{\left(p_\theta + \frac{1}{4\pi}h\right)\left(p_\theta - \frac{1}{4\pi}h\right)}{r^2}\right) + F(r). \quad (108)$$

This gives the equations of motion:

$$\left.\begin{aligned} \dot{r} &= [r, H] = \frac{1}{m}p_r, \\ \dot{p}_\theta &= [p_\theta, H] = 0, \\ \dot{\Theta} &= [\Theta, H] = \frac{p_\theta}{mr^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (109)$$

Let $p_r = m\dot{r}$ and $mr^2\dot{\Theta} = p_\theta =$ constant, as in classical theory. By applying these transformations to the case of the hydrogen atom, Dirac finds a solution for r of the form: †

$$r^{-1} = a_0 + a_1 e^{i\Theta} + a_2 e^{-i\Theta}. \quad . \quad . \quad . \quad . \quad (110)$$

It would be necessary to revert to the co-ordinates x, y in order to deduce the frequencies from this; the following remarks are of great importance for these calculations.

* This means that the transformation is not applicable to the problem of the rotator, where x and y are supposed to satisfy the condition

$$x^2 + y^2 = a^2.$$

† Dirac, *Proc. Roy. Soc.*, A, Vol. **110** (1926), pp. 561–79. I have merely sketched Dirac's methods here, and refer to the original paper for the details of the exposition. The hydrogen atom has also been dealt with by Pauli (*Zts. f. Phys.*, Vol. **36** (1926), p. 336).

By formula (39), we have

$$\frac{2\pi i}{h}(p_\theta \cdot e^{i\alpha\Theta} - e^{i\alpha\Theta} \cdot p_\theta) = [e^{i\alpha\Theta}, p_\theta] = i\alpha e^{i\alpha\Theta},$$

so that

$$e^{i\alpha\Theta} \cdot p_\theta = \left(p_\theta - \alpha \frac{h}{2\pi}\right) e^{i\alpha\Theta}. \quad . \quad . \quad . \quad . \quad . \quad (111)$$

More generally,

$$e^{i\alpha\Theta} f(\Theta, p_\theta) = f\left(\Theta, p_\theta - \alpha \frac{h}{2\pi}\right) e^{i\alpha\Theta}. \quad . \quad . \quad (112)$$

This formula is proved as follows: if it is true for two functions f_1 and f_2 we verify that it is also true for $f_1 \cdot f_2$ and $f_1 + f_2$.

Now the relation holds for $f_1 = \Theta$ and $f_2 = p_\theta$; it is therefore valid for every product of the form $\Theta^n p_\theta^m$ and for every function expanded in powers of Θ and p_θ.

By applying (107) and (110), the x's and y's may be expanded in powers of $e^{i\alpha\Theta}$, where α is a whole number.

We can find the frequencies by setting

$$\frac{d}{dt} e^{i\alpha\Theta} = 2\pi i \nu_\alpha e^{i\alpha\Theta}.$$

The first member is calculated from formula (44):

$$\frac{d}{dt} e^{i\alpha\Theta} = [e^{i\alpha\Theta}, H] = \frac{2\pi i}{h}\{H(p_\theta) e^{i\alpha\Theta} - e^{i\alpha\Theta} \cdot H(p_\theta)\}$$

$$= \frac{2\pi i}{h}\left\{H(p_\theta) - H\left(p_\theta - \alpha \frac{h}{2\pi}\right)\right\} e^{i\alpha\Theta}$$

by applying (112).

Comparing the two formulæ above, we obtain

$$h\nu_\alpha = H(p_\theta) - H\left(p_\theta - \alpha \frac{h}{2\pi}\right). \quad . \quad . \quad . \quad (113)$$

Dirac uses this proof, which in short consists of starting from formula (44) in order to obtain Bohr's frequency-condition (34). He considers that p_θ may be represented by a set of constant values of the form $(n + \beta)\frac{h}{2\pi}$, which gives a set of frequencies

$$h\nu_{n,\, n-\alpha} = H\left((n + \beta)\frac{h}{2\pi}\right) - H\left((n + \beta - \alpha)\frac{h}{2\pi}\right).$$

For the hydrogen atom, Balmer's series is obtained in this way after somewhat intricate calculations. To me it seemed of interest to indicate how Dirac, starting from very general ideas, arrives at the theory of matrices as a particular case.

13. **The Use of Matrices in Mathematics, and their**

Applications.—Born also set himself the problem of finding to what more general class of mathematical quantities matrices belong. We must recall how infinite matrices are usually introduced. We consider a vector x in space of an infinite number of dimensions, and we represent it by its components $x_1, x_2, \ldots, x_n, \ldots$, with the condition that $\Sigma_n |x_n|^2$ converges; the vector is said to be equal to unity if the limit of this sum is 1.

A linear relation between two vectors x and y is written

$$y_m = \sum_n a_{mn} x_n, \qquad \ldots\ldots \quad (114)$$

and we are led to introduce a table of coefficients a_{mn}, that is to say, a matrix a. By means of this matrix we can also construct a bilinear form

$$A(x, z) = \sum_n \sum_m a_{nm} x_n z_m \qquad \ldots\ldots \quad (115)$$

by using the components of the two infinite vectors x and z. If the matrix is of the Hermite type, we shall obtain a real quadratic form by the operation

$$A(x, x^*) = \sum_n \sum_m a_{nm} x_n x_m^*. \qquad \ldots \quad (115a)$$

It is by means of these expressions that a *bounded* matrix is defined; this property may be expressed by Pringsheim's condition:

$$\left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} x_n y_m \right| \leqslant M, \text{ where } x \text{ and } y \text{ are unit vectors.} \quad (116)$$

It may also be exhibited as follows:

The coefficients a_{ns} form an infinite vector (a_s). If we form

$$z_n = \sum_{s=1}^{s=\infty} a_{ns} x_s, \qquad \ldots\ldots \quad (117)$$

where x is a unit vector, the z_n's exist for all values of n and are the components of a vector z for which

$$|z| \leqslant M. \qquad \ldots\ldots \quad (118)$$

It appears that these conditions are normally fulfilled by our matrices *; in particular, the sum of the terms of a row (or of a column) serves to represent the co-ordinate $q_n(t)$ of the state of energy W_n (equations 56, 57, 58), and must therefore converge; this corresponds to the relation (117).

* Born repeatedly asserts that matrices are not bounded, but for criterion he takes the fact that the sum of the diagonal terms is not convergent; in this respect, even the matrix **1** would be unbounded! The above conditions have been established by mathematicians; see, for example, F. Riesz, "Les systèmes d'équations linéaires", *Collection Borel* (Paris, 1913); J. Hyslop, "The Theory of Bounded Matrices", *Proc. London Math. Soc.*, Vol. **24** (1925), p. 264.

The investigation of the solution of a given mechanical problem (§ 5) reduces to finding the principal axes of the quadratic form (115*a*) corresponding to the matrix H. The essential part of the mechanical statement is the reduction of H to diagonal form, which means referring the form $H(x, x^*)$ to its principal axes.

The transformation of a quadratic form to its principal axes has been the object of numerous mathematical investigations. Born has studied the case of a disturbed system by this method, and thus obtains the first approximation results that we found (§ 9). In certain cases, transformation to principal axes does not yield a discontinuous series of proper values H_n, but gives a certain number of distinct values, followed by a continuous series.* This case would correspond to atomic models which lead to a line spectrum followed by a continuous spectrum.

14. **Quantum Magnitudes considered as Functional Operators.**—In a recent paper published in collaboration with the American mathematician N. Wiener,† Born attempts to specify the meaning of quantum magnitudes, for the case where representation by matrices fails. He starts from formula (114), in which we are enabled to pass, by means of a matrix, from one set of components x to another, y.

$$y_m = \sum_n q_{mn} x_n. \quad . \quad . \quad . \quad . \quad . \quad (114a)$$

We may suppose that the x_n's represent the amplitudes of vibrations of frequencies $\frac{1}{h} W_n$, the whole series constituting a motion

$$x(t) = \sum_n x_n e^{\frac{2\pi i}{h} W_n t}, \quad . \quad . \quad . \quad . \quad . \quad (119)$$

and similarly

$$y(t) = \sum_m y_m e^{\frac{2\pi i}{h} W_m t}.$$

The atomic system represented by the matrix $\{q_{mn}\}$ enables a relation to be established between the initial motion $x(t)$ and the final motion $y(t)$.

We can put this transformation in another form; taking a function

$$q(t, s) = \sum_{m,n} q_{mn} e^{\frac{2\pi i}{h}(W_m t - W_n s)}, \quad . \quad . \quad . \quad (120)$$

the passage from x to y is represented by

$$y(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} q(t, s)\, x(s)\, ds. \quad . \quad . \quad (121)$$

* Hellinger, *Crelle's Journal*, Vol. **136** (1910), p. 1; B. II, p. 590.

† *Zts. f. Phys.*, Vol. **36** (1926), pp. 174–87, and *Journ. Math. Phys. Mass.*, Vol. **5** (1926), p. 84.

By means of the matrix $\{q_{mn}\}$ and the system of values W_m, we have formed a function $q(t, s)$ which enables us to obtain the integral relation (121).

We shall say that we apply the *functional operator* $\mathbf{q}$ to the function $x(t)$, and we shall write the transformation symbolically,

$$y(t) = \mathbf{q}x(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

Instead of matrices, we are led to consider *operators*, which may be of the integral type (121), the function $q(t, s)$ being completely arbitrary, but which may be of quite different type: these operators represent the passage from one function x to another, y, according to the very general scheme (122). We assume only that these operators are linear, i.e. such that

$$\mathbf{q}(x_1 + x_2) = \mathbf{q}x_1 + \mathbf{q}x_2.$$

This idea of a functional operator possesses great flexibility; any ordinary function or quantity may be considered as an operator, the expression (122) then reducing to ordinary multiplication; the unit operator $\mathbf{1}$ is that which leaves the function $x(t)$ unaltered. We are already accustomed to the manipulation of the differential and integral operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$, $\int \ldots dt$, and we have frequently written down symbolic operations (determinants, for example) wherein such notation figures. The product $\mathbf{pq}$ of two operators represents the successive application of two transformations, and we cannot assert *a priori* that $\mathbf{pq}$ leads to the same result as $\mathbf{qp}$. The non-commutability of multiplication then appears very natural.

Dirac's quantum numbers are therefore supposed to represent operators, and for these we assume that the fundamental postulates (11), (13), and (14) are valid.

Born has investigated the part played by the operator $D = \frac{\partial}{\partial t}$, and finds that it is not commutable with an operator $\mathbf{q}$ of the integral type (121). We have, in fact,

$$D\mathbf{q}x(t) = \frac{d}{dt}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} q(t, s)x(s)\,ds$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\frac{\partial q(t, s)}{\partial t}x(s)\,ds, \quad . \quad . \quad . \quad (123)$$

$$\mathbf{q}Dx(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} q(t, s)\frac{\partial x}{\partial s}\,ds$$

$$= -\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\frac{\partial q(t, s)}{\partial s}x(s)\,ds, \quad . \quad . \quad (124)$$

the latter formula being obtained by integration by parts, with the assumption that the residue $\frac{1}{2T}[q(t,s)x(s)]_{-T}^{+T}$ is negligible. The two transformations $D\mathbf{q}$ and $\mathbf{q}D$ are therefore of the integral type (121).

Let us consider, for example, the function of transformation (120) derived from a matrix $\{q_{mn}\}$; we can form the functions which correspond to the two transformations above:

$$(Dq)(t,s) = \sum_{m,n} \frac{2\pi i}{h} W_m q_{mn} e^{\frac{2\pi i}{h}(W_m t - W_n s)};$$

$$(qD)(t,s) = \sum_{m,n} \frac{2\pi i}{h} W_n q_{mn} e^{\frac{2\pi i}{h}(W_m t - W_n s)}. \quad . \quad (125)$$

The operator D corresponds to the matrix $\left\{\frac{2\pi i}{h} W_n\right\}$, and the transformation $D\mathbf{q} - \mathbf{q}D$ is associated with the matrix

$$\dot{\mathrm{q}} = \frac{2\pi i}{h}(\mathrm{Wq} - \mathrm{qW}). \quad . \quad . \quad . \quad . \quad (126)$$

These results hold for an operator which can be represented in the form (120) or (121); we shall generalize them by assuming the formula

$$\dot{\mathbf{q}} = D\mathbf{q} - \mathbf{q}D \quad . \quad . \quad . \quad . \quad . \quad (127)$$

as a fundamental definition.

The equations of an atomic system will then appear in the same form as in the theory of matrices, and we shall transcribe our formulæ (33*a*) and (44), or rather (67) and (70):

$$[\mathbf{q}_i, \mathbf{q}_k] = 0; \quad [\mathbf{p}_i, \mathbf{p}_k] = 0; \quad [\mathbf{q}_i, \mathbf{p}_k] = \delta_{ik} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases}. \quad (128)$$

$$\dot{\mathbf{q}}_k = [\mathbf{q}_k, \mathbf{H}] = \frac{\partial \mathbf{H}}{\partial \mathbf{p}_k}; \quad \dot{\mathbf{p}}_k = [\mathbf{p}_k, \mathbf{H}] = -\frac{\partial \mathbf{H}}{\partial \mathbf{q}_k}. \quad (129)$$

The operator D is therefore in general equated to the operator $\frac{2\pi i}{h}\mathbf{H}$.

If we apply the transformation D to a harmonic function $e^{i\omega t}$, we have

$$D^n e^{i\omega t} = (i\omega)^n e^{i\omega t}, \text{ whence } D = i\omega;$$

this relation is obvious if n is an integer, and it can be proved true for any n whatever. Just as we defined a function of matrices,* we can speak of a function $\mathbf{F}(\mathbf{q})$ of an operator, and write down a function like $\mathbf{F}(\mathbf{D})$ symbolically.

* The function represents an expansion in power series.

For any problem, Born and Wiener then investigate solutions of the form

$$\mathbf{q} = \mathbf{u}(t)\mathbf{F}(\mathbf{D}),$$

and show that the Hermite condition may be written:

$$\mathbf{u}(t)\mathbf{F}(\mathbf{D}) = \mathbf{F}^*(\mathbf{D})\mathbf{u}^*(-t), \quad . \quad . \quad . \quad (130)$$

where the asterisk still indicates the conjugate imaginary. Once the solution is obtained in the form of an operator, the value of a co-ordinate is deduced by forming

$$q(t, W_n) = e^{-\frac{2\pi i}{h}W_n t}\, \mathrm{q} e^{\frac{2\pi i}{h}W_n t}. \quad . \quad . \quad . \quad (131)$$

This operation corresponds to that which consists in taking the sum of the terms of a row of a matrix q; it gives the expansion of the co-ordinate q for the atom which is in the state of energy W_n.

I have tried to sketch this latter form of the theory in broad outline, for it apparently enables us to interpret Dirac's mysterious quantum numbers. Born and Wiener exhibit the flexibility of this method, which may be applied to non-periodic problems (e.g. uniform motion) for which the matrix representation fails.

15. **Conclusion.**—In this article I have attempted to summarize a series of papers which mark a very clear evolution in the theory of atomic mechanics. I have not been able to take account of all the questions raised by the different writers, and I shall rapidly mention the most important of these.

The law of conservation of moment of momentum appears in a form very like that of classical mechanics.* The connexion with electromagnetism (B. I, p. 884) and the application of statistical mechanics (B. II, p. 606) have also been sketched.

Lanczos has indicated an analogy between the theory of matrices and that of integral equations, but I preferred to set forth Born and Wiener's exposition.

These various attempts are interesting, after all, in that they exhibit, in as coherent a form as possible, a theory which arranges the old quantum conditions and the principles of selection and correspondence in a logical whole. As regards results in practice, emphasis must be laid on the deduction of Kramers' dispersion formulæ, as well as of certain rules about multiplets.† Born indicates the possibility of logically accounting for the laws of the anomalous Zeeman effect, and Dirac has published a deduction of these very

* Dirac's transformation, formulæ (107), (109), is the most simple example of this. See B. II, p. 595, and the recent paper by L. Mensing, *Zts. f. Phys.*, Vol. **36** (1926), p. 814.

† B. II, pp. 600–5.

laws.* By studying the effect of radiation on a free electron, he obtains the Compton radiation also, i.e. a result which is usually derived from a theory of light quanta. I have myself published several formulæ on rotation spectra, and intend to return to this point later. It appears that the field opened up is a fertile one, and that interesting glimpses of the mysteries of atomic mechanics may be expected from it.

It must also be emphasized that this theory can be reconciled with Schrödinger's very interesting suggestions † which originate from L. de Broglie's ideas. Schrödinger has shown very simply that his theory gives a clear visualization of the Born and Wiener operators.

* *Proc. Roy. Soc.*, A, Vol. **111** (1926), p. 405. Note also an interesting extension of Dirac's methods by G. Wentzel (*Zts. f. Phys.*, Vol. **37** (1926), p. 80).

† *Ann. der Phys.*, Vol. **79** (1926), p. 361 and p. 489. [For additional references see p. 55.]

The Principles of the New Wave Mechanics

BY

LOUIS DE BROGLIE

(*Le Journal de Physique et le Radium*, Vol. **7** (1926), p. 321)

Summary.—The object of the paper is to expound the general principles of the new wave conception of mechanics suggested by the author and confirmed by the recent work of Schrödinger. The author has first sought to show how, in the particular case of constant fields, Schrödinger's results extend and complete his own: he then investigates whether the same ideas can be extended to variable fields and to the dynamics of systems. Finally, he summarizes the method by which Schrödinger has been able to reduce the quantum mechanics of Heisenberg, Born, and Jordan to the new wave mechanics, and he rapidly indicates, in conclusion, the problems raised by the attempt to reconcile the electromagnetic theory with the whole of these conceptions.

I. The Wave Dynamics of a Material Particle in a Constant Field

1. **The Object of the Exposition.**—Under pressure from the results of experiment, physicists have been obliged to admit that the old dynamics, even when enlarged by relativistic ideas, could not interpret phenomena involving quanta. To-day it appears necessary to create a new mechanics closely connected with the theory of waves. This is the idea which I have been endeavouring to work out for some years,* and which the recent beautiful work of M. Schrödinger† has rounded off and extended. I shall attempt to summarize the present state of the question in broad outline.

* See especially *Thèse de doctorat* (Masson, 1924), and *J. Phys.*, Vol. **1** (1926), pp. 1–6.

† *Ann. der Phys.*, Vol. **79** (1926), pp. 361, 489, and 734. I shall designate these three ‡ memoirs by the letters A, B, C.

‡ [Additional references to papers by Schrödinger: *Die Naturwiss.*, Vol. **14**, Part 28 (1926), p. 664; *Ann. der Phys.*, Vol. **80** (1926), p. 437; *Ann. der Phys.*, Vol. **81** (1926), p. 109; *Ann. der Phys.*, Vol. **82** (1927), p. 170; *Ann. der Phys.*, Vol. **82** (1927), p. 178; *Ann. der Phys.*, Vol. **83** (1927), p. 186. An English translation of all nine papers will appear very shortly.]

In what follows, I shall at first only consider the motion of a material particle in a constant field, reserving the extension of the same ideas to the case of variable fields and systems of material particles for the third section.

2. **A Résumé of some Ideas of the Wave Theory.**—Let us consider a region of space whose properties are independent of time. A phenomenon is said to be propagated in waves if the function $u(x, y, z, t)$, which represents it, satisfies the fundamental equation:

$$\nabla^2 u = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

V is a quantity called the " velocity of propagation ", which varies from point to point but which is independent of time. Note that V may be imaginary ($V^2<0$). The quotient, n, of c, the velocity of propagation of light in free space, by the velocity V, often goes by the name of index of refraction. Evidently it comes to the same thing to be given V or n as a function of the co-ordinates.

In order to interpret the experimental facts, physicists have been led to consider especially the following type of solution of equation (1):

$$u(x, y, z, t) = A(x, y, z) \cos 2\pi\nu[t - \Psi(x, y, z)], \quad (2)$$

where ν is a constant called the frequency of the wave, and A a function representing the amplitude of the wave at each point: the argument of the cosine is the " phase " of the wave. This solution may also be considered as the real part of the expression

$$u(x, y, z, t) = C\, e^{2\pi i\nu t}\, e^{2\pi i\phi}, \quad . \quad . \quad . \quad . \quad (3)$$

C being a (real) constant, and ϕ a function, generally imaginary, of x, y, z. If $\phi = a + ib$, we obviously have at once:

$$\Psi = -\frac{a}{\nu}, \quad A = C\, e^{-2\pi b}. \quad . \quad . \quad . \quad . \quad (4)$$

When we confine ourselves to the consideration of sinusoidal solutions of type (2), the wave equation takes the simple form, in which time no longer appears:

$$\nabla^2 u + \frac{4\pi^2\nu^2}{V^2}u = \nabla^2 u + \frac{4\pi^2\nu^2}{c^2}n^2 u = 0. \quad . \quad . \quad . \quad (5)$$

The phase of a sinusoidal wave is the same at a given time on each surface of the family:

$$\Psi(x, y, z) = C', \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where C' is a constant. These are the surfaces of equal phase, and, as the time varies, the values of the phase progress in space, passing

from one surface to the next. The curves orthogonal to the surfaces $\Psi = C'$ are, by definition, called the rays of the wave, and we shall call " phase velocity " the velocity with which we must, at any point, travel along the ray in order to keep to a given value of the phase. If dr is the element of length measured along the ray, the phase velocity is easily found to be

$$\mathcal{V} = \frac{1}{\partial\Psi/\partial r} = \left[\Sigma\left(\frac{\partial\Psi}{\partial x}\right)^2\right]^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

3. **Geometrical Optics.**—We now come to an important question: does a simple relation exist between the two velocities V and $\mathcal{V}$ of equations (1) and (7)?

In order to see, we shall take the sinusoidal solution in the form (3) and substitute in (5), which becomes

$$-4\pi^2\Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 + 2\pi i\nabla^2\phi + \frac{4\pi^2\nu^2}{V^2} = 0. \quad . \quad . \quad (8)$$

If the second differential coefficients of ϕ are very small compared to the sum of the squares of the first differential coefficients, the function $\phi(x, y, z)$ will approximately satisfy the relation

$$\Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 = \frac{1}{\lambda^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where, by definition, we put

$$\frac{V}{\nu} = \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Further, if V is real, ϕ is real also, and we have, by (9), (4), and (7),

$$\Psi = -\frac{\phi}{\nu}, \quad \Sigma\left(\frac{\partial\Psi}{\partial x}\right)^2 = \frac{1}{\mathcal{V}^2} = \frac{1}{V^2}. \quad . \quad . \quad . \quad (11)$$

The velocities V and $\mathcal{V}$ may therefore be taken as equal, and the approximate solution obtained is:

$$u(x, y, z, t) = C\cos 2\pi\nu\left[t - \int\frac{dr}{V}\right], \quad . \quad . \quad (12)$$

where the integral is taken along the ray passing through the point M, whose co-ordinates are (x, y, z), from an equiphase surface chosen as origin, up to M.

Thus, when equation (9) is valid, we can employ the methods of geometrical optics for the study of waves. Let us briefly recall some of these methods. Knowing one equiphase surface, we can construct two others infinitely close to it, by describing round each point M of the given surface a sphere of radius $\epsilon V(M)$, where ϵ is a very small

constant, and taking the two-sheeted enveloping surface of these small spheres. The surfaces thus obtained are in fact those on which, at times $t - \epsilon$ and $t + \epsilon$, the value of the phase is that which occurs on the given surface at time t. The family of equiphase surfaces is thus constructed step by step, and at the same time the rays are determined as limits of broken lines of infinitesimal length. The phase surfaces are then said to be determined by the method of wave envelopes.

Geometrical optics rests on a fundamental postulate called " Fermat's principle ", according to which every ray passing through two points A and B in space is such that the curvilinear integral

$$\int_A^B \frac{dr}{V}$$

is a minimum: that is,

$$\delta \int_A^B \frac{dr}{V} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In the language of the theory of waves, we may say that the time taken by the phase to travel from A to B is a minimum along the ray. The construction of the wave envelopes makes Fermat's principle almost self-evident: thus, in the eyes of Fresnel's followers, Fermat's principle has been lowered from the rank of a postulate to that of a theorem valid under the same conditions as equation (9).

4. **Limits to the Application of Geometrical Optics.**—The methods of geometrical optics are applicable when equation (9) is valid, and we have seen that this requires the second differential coefficients of ϕ to be small compared with the sum of the squares of the first differential coefficients.

Whenever, by applying the methods of geometrical optics, we obtain a function ϕ which does not satisfy this condition we are certain to be on the wrong road.

The quantity λ defined by (10) we shall call wave-length. We must write down the condition that the second differential coefficients are small compared with $1/\lambda^2$. If dl is an element of length in a direction which makes an angle θ with the direction of the ray at the point considered, it follows, from (10) and (11), that

$$\frac{\partial \phi}{\partial l} = -\nu \frac{\partial \Psi}{\partial l} = -\frac{\nu \cos\theta}{V}, \quad \frac{\partial^2 \phi}{\partial l^2} = \frac{\nu \cos\theta}{V^2} \frac{\partial V}{\partial l} = \frac{1}{\lambda} \frac{\cos\theta}{V} \frac{\partial V}{\partial l},$$

and we must have, for all values of θ,

$$\cos\theta \cdot \frac{1}{V} \frac{\partial V}{\partial l} \cdot \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

small in comparison with unity.

Hence we conclude that the methods of geometrical optics are justified if the relative variation of the function V over a small length of the order of λ is very small.

In particular, if geometrical optics leads us to a curved ray contained in a region whose dimensions are of the order of the wave-length, we can be sure that the method used to obtain this result is wrong, since equation (9) cannot hold.

5. **Groups of Waves.**—We may suppose that the function V does not depend on the co-ordinates only, but also on the frequency, considered as a variable parameter: we then say that there is dispersion. This is notably the case when the function u has in reality to satisfy the equation of propagation:

$$\nabla^2 u = pu + q\frac{\partial^2 u}{\partial t^2}, \quad \ldots\ldots \quad (15)$$

where p and q are real functions of the co-ordinates, since, for sinusoidal solutions, this equation is equivalent to (1) if we put

$$V = \left(q - \frac{p}{4\pi^2\nu^2}\right)^{-\frac{1}{2}}. \quad \ldots\ldots \quad (16)$$

Let us now imagine a series of waves of nearly equal frequencies being propagated under these conditions, supposing that we can rely on geometrical optics.

Since the velocities of propagation are not quite the same, the resulting amplitude is propagated along the rays with velocity U different from V. This is the " group-velocity ", the value of which, by a familiar piece of reasoning, is *

$$U = \left(\frac{\partial\left(\frac{\nu}{V}\right)}{\partial\nu}\right)^{-1} \quad \ldots\ldots \quad (17)$$

6. **The Wave Mechanics.**—As opposed to the new wave mechanics, I shall use the names "old mechanics" or "geometrical mechanics" for (1) the classical Newtonian mechanics, (2) the mechanics of relativity. I consider the second an improvement on the first, although, as we know, they coincide in every case where the velocity of the moving body is small enough for $\left(\frac{v}{c}\right)^2$ to be considered negligible as compared with unity. At present, however, even Einstein's dynamics appears to me to be only an approximation compared to a more general theory which is to the old dynamics what the wave theory of light is to geometrical optics.

* [See footnote, p. 10.]

7. **Some Definitions of the Old Mechanics.**—We learn from the old mechanics that, in the case of constant fields at present being considered, a certain function of the co-ordinates and of the velocity of the moving body remains invariable as time goes on: it is the energy $W(x, y, z, v)$. In Einstein's dynamics,* we have

$$W = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + F(x, y, z), \quad . \quad . \quad . \quad . \quad (18)$$

where $F(x, y, z)$ is the potential energy of the moving body of proper mass m_0 at the point (x, y, z). To obtain the classical expression for W, we have only to expand $\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ in series and neglect the higher terms:

$$W = m_0 c^2 + \tfrac{1}{2} m_0 v^2 + F(x, y, z). \quad . \quad . \quad . \quad (18')$$

When classical mechanics holds, the constant $m_0 c^2$ is much greater than the terms which follow. In the old textbooks, the name of energy is applied to the variable part of W, the sum of the kinetic and potential energies: in what follows, we shall designate this by E.

Alongside of energy, the old dynamics introduces a vectorial quantity, the momentum $\mathbf{g}$. Leaving aside for the moment the case where the moving body carries an electric charge and moves in a magnetic field, we shall express the momentum as a function of the single variable v by the (relativistic) formula:

$$\mathbf{g} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The corresponding Newtonian equation is:

$$\mathbf{g} = m_0 \mathbf{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19')$$

By eliminating the velocity between the expressions for g and W, g is obtained as a function of the co-ordinates and of W. We have

$$\text{in relativity mechanics, } g = \frac{1}{c}\sqrt{(W - F)^2 - m_0^2 c^4}, \quad (20)$$

$$\text{in classical mechanics, } g = \sqrt{2m_0(E - F)}. \quad . \quad . \quad (20')$$

8. **The Wave associated with the Moving Body.**—The essential idea which has guided me in my previous work has been

* [T. Levi-Civita, *Absolute Differential Calculus*, Chapter XI, § 5.]

that of associating the propagation of a wave with the motion of every moving body. Thus I considered a sinusoidal wave satisfying equation (1), and, on the supposition that the approximate solutions of geometrical optics were valid, I wrote the expression for the wave in the form (12). In order to establish a bond between the wave thus defined and the corresponding mechanical problem, I was led to set:

$$W = h\nu, \quad g = \frac{h\nu}{V}, \qquad \ldots\ldots \quad (21)$$

h being Planck's constant. The first relation is in some measure imposed by the theory of light quanta: the second is derived from the first by considerations of invariance. A consequence of the relations (21) is the fact that the variation of the phase, when we go a distance dr along the ray during time dt, is proportional to the corresponding variation of the Hamiltonian action, for we have

$$2\pi\nu\left(dt - \frac{dr}{V}\right) = \frac{2\pi}{h}(W\,dt - g\,dr). \qquad \ldots \quad (22)$$

In the old mechanics, the form of the trajectories in a constant field is determined by Maupertuis' principle:

$$\delta \int_A^B g\,dr = 0. \qquad \ldots\ldots \quad (23)$$

Comparing (23) and (13), and remembering the relations (21), we see that the possible trajectories of the moving body coincide with the rays of the associated wave. Owing to the identity of Fermat's principle with the principle of least action, the dynamical problem is reduced to the study of the propagation of sinusoidal waves.

According to (21), we must, in order to obtain the equation of propagation of the associated wave in a given field, replace $\frac{1}{V}$ in equation (1) by $\frac{g(x, y, z, W)}{h\nu}$.

Thus, according as we adopt Einstein's or Newton's mechanics, we obtain:

$$\nabla^2 u + \frac{4\pi^2}{h^2 c^2}\left[(W - F)^2 - m_0{}^2 c^4\right] u = 0, \qquad \ldots \quad (24)$$

or

$$\nabla^2 u + \frac{8\pi^2}{h^2} m_0 (E - F) u = 0. \qquad \ldots \quad (24')$$

Note that the velocity of propagation depends on the frequency ν (through the energy); there is dispersion. We shall also see later that in the final analysis the true equation of propagation is not of type (1), but of a more complicated type.

We shall verify that geometrical mechanics is derived from (24) or from (24′) with the same reservations as when geometrical optics is derived from (1).*

Let us write the sinusoidal solution in the form (3), putting

$$\phi(x, y, z) = \frac{1}{h} S(x, y, z). \quad . \quad . \quad . \quad . \quad (25)$$

Substituting in (24) or (24′) and supposing that the second differential coefficients of ϕ are small compared with the sum of the squares of the first differential coefficients, we arrive at one or other of the relations:

$$\Sigma \left(\frac{\partial S}{\partial x}\right)^2 = \frac{(W - F)^2}{c^2} - m_0^2 c^2, \quad . \quad . \quad . \quad (26)$$

$$\frac{1}{2m_0} \Sigma \left(\frac{\partial S}{\partial x}\right)^2 + F = E. \quad . \quad . \quad . \quad . \quad . \quad (26')$$

(26′) is the classical equation of Jacobi † for constant fields; (26) is its relativistic form. S is the function of action, and the wave can be written in the form analogous to (12):

$$u(x, y, z, t) = C \cos \frac{2\pi}{h} (Wt - S). \quad . \quad . \quad (27)$$

It is interesting to note that the condition as to the differential coefficients of ϕ would always be fulfilled if it were possible to suppose h infinitely small, for the second differential coefficients of ϕ are proportional to $1/h$, while the square of the first differential coefficients is proportional to $1/h^2$. Hence we shall conclude that the wave mechanics coincides with the old mechanics when h tends towards zero, a result which was to be expected *a priori*.

We shall now indicate a formal method, the importance of which will appear directly, which enables us to get back from the equation (26) or (26′) to the equation of propagation (24) or (24′). We know that the differential coefficients of S with respect to the co-ordinates are the conjugate Lagrangian momenta; if, in (26) or (26′), we replace $\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}$ by p_x, p_y, p_z, we obtain the equation of energy, which may be written

$$f(x, y, z, p_x, p_y, p_z) = 0. \quad . \quad . \quad . \quad . \quad (28)$$

* Cf. Léon Brillouin, *C. R.*, Vol. **183** (1926), p. 270; G. Wentzel, *Zts. f. Phys.*, Vol. **38** (1926), p. 518.

† [Whittaker, *loc. cit.* (p. 12 above), § 142; Birtwistle, *loc. cit.*, p. 58; Born, *loc. cit.*, p. 36.]

In the function f, replace p_x, p_y, p_z, respectively, by the symbols $K\frac{\partial}{\partial x}$, $K\frac{\partial}{\partial y}$, $K\frac{\partial}{\partial z}$, with the notation

$$K = \frac{h}{2\pi i}. \quad \ldots\ldots\ldots \quad (29)$$

An operator is thus obtained: if we apply this operator to the function u and equate the result to zero, we obtain the equations (24) and (24′). For example, starting from (26′) and proceeding as we have just indicated, we obtain the operator:

$$\frac{1}{2m_0} K^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + F - E,$$

and, by equating the result of its operation on u to zero, we end up with (24′).

Similarly, (26) leads to (24).

9. **Schrödinger's important Observation.***—Quantum theories have the world of the atom as their principal field of application. Assuming my conception of mechanics, Schrödinger wondered whether the propagation of the associated wave on the strictly atomic scale could be studied by the methods of geometrical optics. Now, this is not the case, as may be seen from what follows. So far, the methods of geometrical mechanics have been applied to the atom by adding the quantum conditions, to which we shall return. Proceeding in this way, the existence of continuous trajectories in the atomic domain is foreseen: if, then, we assume that dynamics is of an undulatory nature at bottom, the use of the old methods is not justifiable when the wave-length of the wave associated with an electron within the atom is of the order of atomic dimensions. Now the formulæ (10), (21), and (19′) give (relativity corrections being negligible here):

$$\lambda = \frac{h}{g} = \frac{h}{m_0 v}. \quad \ldots\ldots\ldots \quad (30)$$

As the quotient h/m_0 is of order unity in c.g.s. units, λ is of order v^{-1}, and by using Bohr's equations it is easy to verify that λ is comparable with the dimensions of the atom. We may then logically conclude that atomic dynamics cannot use the Lagrange-Hamilton equations, which may be legitimately used in ordinary mechanics and *a fortiori* in celestial mechanics: atomic dynamics must investigate the solutions of the rigorous equations (24) and (24′) directly. This important observation of Schrödinger has definitely fixed the character of the new mechanics.

* B, p. 149 and following.

10. **The Material Particle.**—Let us return to the case where the equation (26) or (26′) is valid. The rays of the associated wave are the possible trajectories of the moving body. But in a *given* motion, one of the rays is of particular physical importance, namely, the ray which is actually described by the moving body. The associated wave, in so far as we have imagined it up to the present, does not enable us to understand what distinguishes this ray from the others. In order to get some light on this point it is necessary, as I pointed out in my thesis,* to consider not only a monochromatic associated wave, but a group of associated waves with frequencies very close to each other.

The resulting amplitude may then be thought of as exhibiting a very pronounced maximum at one point; this maximum would be the material particle. This way of looking at the matter is made probable by the following fact: according to the old mechanics, the velocity of the moving body is the reciprocal of the differential coefficient of the function $g(x, y, z, W)$ of equations (20) and (20′) with respect to W. We have:

$$v = \left(\frac{\partial g}{\partial W}\right)^{-1}, \quad \ldots\ldots\ldots \quad (31)$$

and, according to (21), this is equivalent to

$$v = \left(\frac{\partial \frac{\nu}{V}}{\partial \nu}\right)^{-1} \quad \ldots\ldots\ldots \quad (32)$$

(17) then shows us that the velocity of the moving body coincides with the velocity of the group of associated waves.

But what happens if the dynamical phenomenon runs its course in a domain of the order of a wave-length, as in the case of the atom? Schrödinger † thinks that then we can no longer speak of a material particle describing a trajectory: the reason he gives is that the region where the amplitude of a group is near its maximum has necessarily an extent of several wave-lengths. The material particle would no longer be a point according to the wave-length scale, and in the atom we could not speak of the position or orbit of an electron. Yet an atom whose dimensions are of the order of 10^{-8} cm. can absorb a quantum of ultra-violet radiation whose wave-length is more than a thousand times as great (the photo-electric effect): this would make me inclined to believe rather that the region where energy is localized must be a point, even on the wave-length scale. This is a difficult problem, which we shall meet with in various forms farther on.

* *Thèse*, p. 16.

† B, pp. 507–8.

II.—Stability of Motion in Constant Fields

11. **The Old Quantum Conditions of Stability.**—The study of the phenomena of quanta has obliged physicists to introduce into mechanics an idea which has for long appeared extremely strange: among the infinity of periodic motions that a material particle can have in a constant field, only certain privileged motions, which form an enumerable series like the series of whole numbers, are stable.

The introduction of the idea of associated waves has solved this riddle.* If, in fact, we suppose that the conditions of application of geometrical optics are realized, the associated wave can be expressed in the form (27), and, in order that the function u may be uniquely determined at every point of space, we must have

$$\int_C dS = nh \qquad (n \text{ an integer}), \qquad . \; . \; . \quad (33)$$

where the integral is taken along any closed curve C.

This is the general form of the quantum conditions enunciated by Einstein in 1917.† The appearance of integers in the dynamical formulæ ceases to be mysterious, and becomes as natural as their occurrence in the theory of vibrating strings or of wireless antennæ. We can express (33) otherwise by saying that all the cyclic periods of the integral of action must be multiples of the constant h.

It is to be noted that quantization presents itself in somewhat different forms,‡ according as we consider the periods obtained by variation of a cyclic variable like an azimuth or of a non-cyclic variable like a radius vector. For example, in the field of a central attracting force, azimuthal quantization leads to a condition of the form:

$$\int_0^{2\pi} \frac{\partial S}{\partial \theta}\, d\theta = n_1 h, \qquad . \; . \; . \; . \; . \quad (34)$$

while radial quantization is expressed by

$$2\int_{\rho_0}^{\rho_1} \frac{\partial S}{\partial \rho}\, d\rho = n_2 h, \qquad . \; . \; . \; . \; . \quad (35)$$

where ρ_0 and ρ_1 are the values of ρ which give maximum and minimum values of the radius vector (branch-points of the function S) and which correspond to the circles on which the radial momentum vanishes.

The conditions summarized by (33) constitute the natural form in which quantum stability must be stated in the language of the old

* *Thèse*, Chapter III. † *Ber. deutsch. phys. Ges.* (1917), p. 82.

‡ [A. Sommerfeld, *Atombau und Spektrallinien*, 4th edition, p. 140.]

mechanics, but now they are no longer to be considered as anything more than approximations of limited value. In particular, a condition of radial quantization such as (35) can *never* be exact, for it can be shown that in the neighbourhood of the circles of radius ρ_0 and ρ_1 geometrical optics is certainly not valid.

12. **The New Conditions of Stability.**—The function u of the equations (24), (24′) must from its physical nature be everywhere one-valued and continuous, and must vanish at infinity. We must therefore find out for which values of the constants W or E the equations (24), (24′) possess solutions of the above type. It is in this form that Schrödinger states the conditions of stability which are valid in every case. He thus determines the energy of the motions, or rather the frequency of the stable waves.*

We shall consider equation (24′), which is the only one treated by Schrödinger, more closely. I assert that: (1) there exists a series of values E for which (24′) possesses a solution everywhere one-valued and finite; (2) the functions u thus obtained form a system of orthogonal functions in general.

Let us set

$$\mu = \frac{2\pi}{h^2} m_0 E, \quad \frac{8\pi^2 m_0}{h^2} F(x, y, z) = R(x, y, z). \quad . \quad (36)$$

If M is the point whose co-ordinates are (x, y, z), (24′) may be written:

$$\nabla^2 u(M) + [4\pi\mu - R(M)]\, u(M) = 0. \quad . \quad . \quad (37)$$

If $u(M)$ is everywhere one-valued and continuous and vanishes at infinity, the solution of (37) can be reduced to that of a homogeneous integral equation of Fredholm's type:

$$u(M) = \mu \int K(M, P) u(P)\, dv_P, \quad . \quad . \quad . \quad (38)$$

where $K(M, P)$ is a "kernel" conveniently chosen; dv_P denotes an element of volume containing the point P and the integral is extended throughout all space. It is known that such an integral equation possesses non-zero solutions for certain values only of the constant μ. In general to each of these particular values μ_i there corresponds a single function u_i. The μ_i's are the "proper values", and the u_i's the "proper functions" † of equations (37) and (38).

It now remains for us to show that the functions u_i are in general orthogonal, that is to say, they satisfy the relations

$$\int u_i(P)\, u_j(P)\, dv_P = 0, \quad i \neq j. \quad . \quad . \quad . \quad (39)$$

* A and B.

† [French: *constantes fondamentales, fonctions fondamentales.*]

As the functions u_i and u_j are solutions of (37) for the values μ_i and μ_j of the constant μ, the following relation is easily deduced:

$$u_j \nabla^2 u_i - u_i \nabla^2 u_j = \Sigma \frac{\partial}{\partial x}\left(u_j \frac{\partial u_i}{\partial x} - u_i \frac{\partial u_j}{\partial x}\right) = 4\pi (\mu_j - \mu_i) u_i u_j. \quad (40)$$

Integrating throughout space, we obtain, since the surface integrals vanish at infinity:

$$(\mu_i - \mu_j)\int u_i(P)\, u_j(P)\, dv_P = 0, \quad . \quad . \quad . \quad (41)$$

and this is equivalent to (39) if to every μ_i there corresponds a single function u_i.

As the u_i's are evidently undefined to the extent of a multiplicative constant, they can be made normal by choosing the constants so as to have

$$\int u_i^2(P)\, dv_P = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

for all values of i.

I shall dwell no longer on these generalities.* What I have just said is sufficient to show how the quantization problem is reduced to the investigation of certain proper values. The determination of the corresponding proper functions presents a certain arbitrary character in the cases where several of these functions correspond to the same value μ_i. These are precisely the cases where, in the mathematical language of the old mechanics, there was said to be " degeneration ".

Schrödinger has studied some remarkable cases. Without having recourse to the theory of integral equations, he determines the proper values and the proper functions by the following artifice: he reduces the equation of propagation to the investigation of an equation of Laplace's type †

$$(a_0 x + b_0) y'' + (a_1 x + b_1) y' + (a_2 x + b_2) y = 0, \quad (43)$$

the solutions of which are expressed as integrals taken along certain curves in the plane of the complex variable in accordance with a classic method.‡ The study of these solutions enables the required values and functions to be determined.

The results thus obtained would deserve a whole paper to them-

* For the theory of integral equations and proper functions, the following works, among others, may be consulted: Goursat, *Traité d'analyse*, Vol. 3 (Gauthier Villars), 1923 edition; Heywood and Fréchet, *L'équation de Fredholm et ses applications* (Hermann), 1913 edition. [Also: M. Bôcher, *Introduction to the study of integral equations* (Cambridge Mathematical Tract); Whittaker and Watson, *Modern Analysis*, Chapter XI; Courant and Hilbert, *Methoden der Math. Physik*, Chapter III.]

† [Gray, Mathews, and MacRobert, *Bessel Functions*, p. 46.]

‡ See Goursat, *Traité d'analyse*, Vol. 2, p. 464.

selves. In order not to make the present memoir too long, I shall refer the reader to Schrödinger's fine papers, confining myself to recalling the two following points:

1. In the case of the *linear oscillator*,* the formula of semi-quanta, previously deduced by Heisenberg from his quantum mechanics, is obtained. This formula, which was suggested by experiment, was not in agreement with the old method of quantization.

2. In the case of the *hydrogen atom*,† if $E > 0$, all the solutions can be accepted; the hyperbolic motions are not quantized. If, on the contrary, $E < 0$, we come back to the levels of energy of Bohr's theory. Thus here the μ_i's form a discrete series of negative values (line spectrum) and a continuous series of positive values (continuous spectrum).

III.—General Equations of Propagation

13. **General Equation of Propagation for the Free Particle.**—In the case of a free material particle, the equation of propagation of the associated wave is

$$\nabla^2 u + \frac{4\pi^2 \nu^2}{c^2}\left[1 - \frac{m_0{}^2 c^4}{h^2 \nu^2}\right] u = 0, \quad . \quad . \quad . \quad (44)$$

from (24), replacing W by $h\nu$.

Comparing this with (5), we see that space exhibits a refractive index

$$n = \sqrt{1 - \frac{m_0{}^2 c^4}{h^2 \nu^2}} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

for the associated wave.

Thus there is dispersion, and as we said in connexion with groups of waves, this leads us to believe that equation (44) is a degenerate form of some more general equation such as (15). To find this equation, we shall first recall the method by which we could get back from the equation of energy in the form (28) to the equations of propagation (24) and (24′). We shall apply the same method, but this time we shall treat the time as a variable analogous to the space variables, as the theory of relativity requires. Let us therefore put:

$$x = x_1, \quad y = x_2, \quad z = x_3, \quad ct = x_4. \quad . \quad . \quad . \quad (46)$$

We learn from relativity dynamics that the conjugate momenta of these four variables are the components of the world-impulse:

$$p_1 = g_x, \quad p_2 = g_y, \quad p_3 = g_z, \quad p_4 = \frac{W}{c}, \quad . \quad . \quad (47)$$

* B, p. 514. † A, *passim*.

where the g's are the components of the momentum, and W the energy.

The equation of conservation of energy can then be written:

$$p_4^2 - p_1^2 - p_2^2 - p_3^2 - m_0^2 c^2 = 0. \quad . \quad . \quad (48)$$

Generalizing the method used for (28), we shall replace each p_i in (48) by $K\frac{\partial}{\partial x_i}$, where K has the value given by (29). Applying the operator thus obtained to the function u, and equating the result to zero, we obtain:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + \frac{4\pi^2}{h^2} m_0^2 c^2 u = 0. \quad . \quad . \quad . \quad (49)$$

Equation (44) is the form that (49) takes when we confine ourselves to the consideration of monochromatic waves of frequency ν, but (49), which is of the form (15), appears to be the general equation of propagation of all the waves of the group associated with the moving body of mass m_0.

In a given system of reference, the free material particle is displaced in a straight line with constant velocity v. If the trajectory is taken as the z axis, it seems probable that the group of associated waves can be represented by the function:

$$u(x, y, z, t) = f(x, y, z - vt)\, e^{2\pi i\nu\left[t - \frac{n}{c}z\right]}, \quad . \quad . \quad (50)$$

where n has the value (45). Substitute in (49) and put:

$$x_0 = x, \quad y_0 = y, \quad z_0 = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t_0 = \frac{t - \frac{v}{c^2}z}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (51)$$

We get the two relations:

$$v = nc, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial y_0^2} + \frac{\partial^2 f}{\partial z_0^2} = 0. \quad . \quad . \quad . \quad . \quad (53)$$

Equation (52) is immediately verified, for, by replacing n by $\frac{v}{c}$ in (45), we get back to the expression for the energy as a function of the velocity. As for equation (53), it expresses the fact that the function f is a harmonic function in the proper system of the moving body, for (51) is the Lorentz transformation. If the moving body has central symmetry in its proper system, the function u, expressed by means of the variables x_0, y_0, z_0, t_0, has for its real part

$$u(x_0, y_0, z_0, t_0) = \frac{C}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \cos 2\pi\nu_0 t_0, \quad . \quad (54)$$

where $\nu_0 = \nu\sqrt{1-n^2}$. With the variables x, y, z, t, the expression for the group of waves would therefore be:

$$u(x, y, z, t) = \frac{C}{\sqrt{x^2 + y^2 + \frac{(z - nct)^2}{1 - n^2}}} \cos 2\pi\nu\left[t - \frac{n}{c}z\right]. \quad (55)$$

The solution (55) is different from the one I have hitherto proposed,* but to me it seems preferable. It will be noted that this solution defines the material point as a singularity which is *strictly* a point.

14. **The Case of the Motion of an Electric Charge in an Electromagnetic Field.**—Following out the same method, we shall investigate the general equation of propagation of the waves associated with the motion of a material particle with charge e in a known electromagnetic field, that is, a field known as a function of the co-ordinates and of the time.

The field is defined by the components of the world-potential which comprises the scalar potential Ψ and the vector potential $\mathbf{a}$. These components are:

$$\phi_1 = -\frac{a_x}{c},\ \phi_2 = -\frac{a_y}{c},\ \phi_3 = -\frac{a_z}{c},\ \phi_4 = \frac{\Psi}{c}. \quad . \quad (56)$$

Between the components of ϕ and those of the world-impulse the invariant relation holds:

$$(p_4 - e\phi_4)^2 - \sum_{i=1,\,2,\,3}(p_i - e\phi_i)^2 = m_0^2 c^2. \quad . \quad . \quad . \quad (57)$$

Let us again replace each p_i by $K\dfrac{\partial}{\partial x_i}$: bearing in mind the Lorentz relation between the potentials,

$$\frac{1}{c}\frac{\partial \Psi}{\partial t} + \operatorname{div}\mathbf{a} = 0, \quad . \quad . \quad . \quad . \quad . \quad (58)$$

we obtain

$$K^2\left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u\right] - 2K\frac{e\Psi}{c^2}\frac{\partial u}{\partial t} - 2K\frac{e}{c}\sum a_x\frac{\partial u}{\partial x} + \left[\frac{e^2}{c^2}(\Psi^2 - a^2) - m_0^2 c^2\right]u = 0. \quad (59)$$

This equation should probably be looked on as regulating the propagation of the waves associated with the movement of the electron in a region where the potentials are known as a function of the co-ordinates and of the time; in precise terms, the functions u which

* See especially *C. R.*, Vol. **180** (1925), p. 498.

represent the associated waves would be the real parts of certain solutions of (59). It is, however, to be observed that (59) contains imaginary terms (those in which K occurs), and this perhaps raises some objections from the physical point of view.

Be that as it may, on putting

$$u = e^{\frac{2\pi i}{h}S}$$

in (59), we obtain the general relativistic form of Jacobi's equation to a first approximation:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - e\Psi\right)^2 - \Sigma\left(\frac{\partial S}{\partial x} + \frac{e}{c}a_x\right)^2 = m_0^2 c^2. \quad . \quad (60)$$

If the fields are constant, equation (59) leads to results which are already known, for we may then suppose that u is periodic, of constant frequency ν.

In an electrostatic field ($\mathbf{a} = 0$), equation (59) takes the form (24) and the space occupied by the field has the refractive index

$$n = \sqrt{\left(1 - \frac{e\Psi}{h\nu}\right)^2 - \frac{m_0^2 c^4}{h^2 \nu^2}} \quad . \quad . \quad . \quad . \quad (61)$$

for the wave.

If there is a magnetic field and if geometrical optics is applicable, we can content ourselves with equation (60): the refractive index, at a point M and in a direction making an angle θ with the vector potential, is then

$$n = c\sqrt{\Sigma\left(\frac{\partial S}{\partial x}\right)^2} \Big/ \frac{\partial S}{\partial t}$$

$$= \frac{c}{h\nu}\left[\frac{ea\cos\theta}{c} \pm \sqrt{\left(\frac{h\nu - e\Psi}{c}\right)^2 - m_0^2 c^2 - \frac{e^2 a^2 \sin^2\theta}{c^2}}\right]. \quad . \quad (62)$$

As a result of the presence of the vector potential, space behaves as if it were anisotropic and doubly-refracting.*

15. **Dynamics of Systems.**—The dynamics of systems of material particles does not seem to present any particular difficulty in principle in the new mechanics when geometrical optics is valid, for we then come back to Jacobi's equation and the old mechanics.

The same does not hold when geometrical optics is no longer applicable. If in this case there still exists in every group of waves a *point*-singularity worthy of the name of material particle, we can easily conceive that each of the groups of waves is propagated according to a law like (59), where the coefficients depend on the position at that time of the singularities in the other groups; but if, in this kind of motion, there are no longer any well-defined material par-

* *Thèse*, p. 39.

ticles, the question becomes obscure. Schrödinger has leanings towards this latter opinion, and, in order to establish the dynamics of systems,* he generalizes my idea of phase-waves: with the motion of the system of N moving bodies he associates a wave in the space of $3N$ dimensions imagined by the classical theories in order to represent the motion of the whole system by the displacement of a single representative point. This wave associated with the system of N particles would thus be a function of $3N$ space-co-ordinates and of the time.

Up to the present I have been unable to accept this point of view; for me the associated waves have physical reality and must be expressed as functions of the three space-co-ordinates and of the time. I cannot dwell further on this difficult question; it is obvious, from the above, that the wave dynamics of systems does not seem to be solidly established as yet.

IV.—The Connexion with Heisenberg's Theory and with Electromagnetism

16. **The Quantum Mechanics of Heisenberg.**†—During the last few years, as a consequence of Heisenberg's profound suggestions, a whole theory of very abstract appearance, known as the quantum mechanics, has been developed: thanks to the work of Born, Jordan, Dirac, Pauli, L. Brillouin, and others, it has taken a remarkable mathematical form and yields interesting results.

All the knowledge that we possess about the interior of atoms comes to us from the study of spectra; starting from this observation, Heisenberg judges it rash to speak of the positions and velocities of electrons within the atom, and prefers to represent the state of the atom by quantities directly connected with the spectral frequencies and intensities observed. Instead of each co-ordinate q_l and moment p_l that appears in Bohr's theory, a table of rows and columns which looks like a determinant is introduced, the general term of which would be:

$$q_l^{ik} e^{2\pi i \nu_{ik} t} \text{ or } p_l^{ik} e^{2\pi i \nu_{ik} t}, \quad . \quad . \quad . \quad . \quad (63)$$

with the additional convention that q_l^{ki} and p_l^{ki} are the conjugate imaginaries of q_l^{ik} and p_l^{ik}. The frequencies ν_{ik} are the emission frequencies of the atom, which may be written in the form:

$$\nu_{ik} = \nu_i - \nu_k, \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

according to the fundamental law of combination of Ritz.

* B, *passim*, especially p. 522 and following pages.

† For the subject of quantum mechanics, the reader is referred to the account of Léon Brillouin in *J. Phys.*, Vol. 7 (1926), p. 135, where a bibliography of the subject will be found. [See p. 19 above.]

The quantities q_l^{ik} are such that the squares of their moduli,

$$| q_l^{ik} |^2 = q_l^{ik} \cdot q_l^{ki}, \quad . \quad . \quad . \quad . \quad . \quad (65)$$

are equal to the intensities of the spectral lines.

In order to be able to use these tables of numbers, we must know the rules of calculation which apply to them. Heisenberg, Born, and Jordan have shown that the rules are those used by mathematicians for algebraic matrices.* Addition and multiplication of these matrices † are defined by the formulæ:

$$(a+b)^{ik} = a^{ik} + b^{ik}, \quad . \quad . \quad . \quad . \quad . \quad (66)$$

$$(a \cdot b)^{ik} = \sum_j a^{ij} \cdot b^{jk}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

It follows from (67) that the order of factors cannot be changed in a product of matrices.

To introduce Planck's constant into quantum mechanics we take as a fundamental postulate that the matrices q_l and p_l satisfy the relations:

$$\left.\begin{aligned} (p_l p_m - p_m p_l)^{ik} &= 0, \\ (q_l q_m - q_m q_l)^{ik} &= 0, \\ (p_l q_m - q_m p_l)^{ik} &= 0, \text{ if } l \neq m, \\ (p_l q_l - q_l p_l)^{ik} &= \begin{cases} 0, \text{ if } i \neq k, \\ \dfrac{h}{2\pi i} \text{ if } i = k. \end{cases} \end{aligned}\right\} \quad . \quad . \quad . \quad (68)$$

In order to determine the p_l's, the q_l's, and the ν_{ik}'s, the quantum mechanics sets down the equations of Hamiltonian appearance:

$$\left.\begin{aligned} \left(\frac{dq_l}{dt}\right)^{ik} &= 2\pi i \nu_{ik} q_l^{ik} = \left(\frac{\partial H}{\partial p_l}\right)^{ik} \\ \left(\frac{dp_l}{dt}\right)^{ik} &= 2\pi i \nu_{ik} p_l^{ik} = \left(-\frac{\partial H}{\partial q_l}\right)^{ik} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (69)$$

In the last term of these equations, H stands for a function of matrices, which is expressed by means of the matrices q_l and p_l in the same way as the energy is expressed by means of the co-ordinates and moments in the corresponding classical problem. For the exact definition of the functions of matrices and of their derivatives, the reader is referred to the original memoirs or to Léon Brillouin's account.‡

* [For matrices, see M. Bôcher, *Introduction to Higher Algebra*; H. W. Turnbull, *Theory of Invariants*.]

† L. Brillouin, *loc. cit.*, p. 137. [P. 22.] ‡ *Loc. cit.*, p. 139 sqq. [P. 24 sqq.]

I shall observe that in fact scarcely any applications of the equations (69) have been made, except in the case of a single moving body, and that there it has seemed necessary to choose for H the expression for the energy in rectangular Cartesian co-ordinates in order to obtain accurate results (Dirac).*

In Heisenberg's theory † it is shown that if f is any function of the q_l's and of the p_l's, the relations (68) involve the following:

$$\left(\frac{\partial f}{\partial q_l}\right)^{ik} = \frac{2\pi i}{h}(p_l f - f p_l)^{ik}; \quad \left(\frac{\partial f}{\partial p_l}\right)^{ik} = \frac{2\pi i}{h}(f q_l - q_l f)^{ik}. \quad (70)$$

Applying these formulæ to the function H and substituting in (69), the equations of quantum mechanics are put in the form:

$$\left.\begin{aligned} \nu^{ik} q_l{}^{ik} &= \frac{1}{h}(Hq_l - q_l H)^{ik} \\ \nu^{ik} p_l{}^{ik} &= \frac{1}{h}(Hp_l - p_l H)^{ik} \end{aligned}\right\} \quad . \quad . \quad . \quad (71)$$

17. **Schrödinger's Interpretation.**—Schrödinger has succeeded in showing, by means of a remarkable transformation,‡ that the equations of Heisenberg's theory can be interpreted naturally in the wave mechanics. To do this, he starts from the definition of " well-ordered " functions. These are functions in which changing the order of the factors is not allowed: thus, for example, the expressions $xy + yx$ and $2xy$ are identical if they are considered as ordinary functions but different in so far as they are considered as well-ordered functions.

Let us then consider a well-ordered function $F(q_l, p_l)$ of the two sets of variables q_l and p_l, and let us further suppose that F is an integral polynomial in p_l. If we replace each p_l in F by the symbol $K\frac{\partial}{\partial q_l}$, where K has the value given by (29), we shall obtain an operator: this is a method which we have already used more than once. Denote the result of the operation on a function u of the variables q_l by $(F \cdot u)$. Finally, suppose that we know a set of functions u_i which are normal and orthogonal, i.e. which satisfy the equations (41) and (42). Schrödinger defines the quantities

$$F^{ik} = \int u_i(P)\,[F \cdot u_k(P)]\,dv_P, \quad . \quad . \quad . \quad (72)$$

where the integral is taken throughout all space, as components of the matrix corresponding to the well-ordered function $F(q_l, p_l)$, and shows that the F^{ik}'s satisfy the rules of calculation (66–67).

* Dirac, *Proc. Roy. Soc.*, Vol. **110** (1926), p. 570 and following pages.

† See L. Brillouin, *loc. cit.*, equations (35) and (39). [P. 28.] ‡ C.

If, in particular, the functions $F = q_l$ and $F = p_l$ are considered, we obtain:

$$\left.\begin{aligned} q_l{}^{ik} &= \int u_i(P)\, u_k(P)\, q_l\, dv_P \\ p_l{}^{ik} &= \int u_i(P)\, K \frac{\partial}{\partial q_l} u_k(P)\, dv_P \end{aligned}\right\} \quad . \quad . \quad . \quad (73)$$

It is easy to verify that with these definitions the relations (68) are satisfied. The equations (69) then take the form (71), which we have to satisfy. There is only one way of doing this, namely, to make a suitable choice of the system of functions u_i, arbitrary up to now. We shall choose the system of the proper functions of equation (24′), or, *what comes to the same thing*, of the equation

$$[H \cdot u] - Eu = 0, \quad . \quad . \quad . \quad . \quad . \quad (74)$$

where H is the expression for the energy in rectangular Cartesian co-ordinates, an expression which is in fact a polynomial in p_l.

The equations (71) are then satisfied; it is sufficient to verify this for the first one. We have:

$$H^{ik} = \int u_i(P)[H \cdot u_k(P)]\, dv_P = E_k \int u_i(P) u_k(P)\, dv_P = \begin{cases} 0 \text{ if } i \neq k \\ E_k \text{ if } i = k \end{cases}. \quad (75)$$

H is therefore a diagonal matrix whose diagonal terms are the E_k's; this is a known result of quantum mechanics. Hence we deduce:

$$(Hq_l - q_l H)^{ik} = \sum_j H^{ij} q_l{}^{jk} - \sum_j q_l{}^{ij} H^{jk} = (E_i - E_k)\, q_l{}^{ik}. \quad (76)$$

Equation (71) is then verified if the ν_{ik}'s obey Bohr's law of frequency:

$$\nu_{ik} = \frac{1}{h}(E_i - E_k). \quad . \quad . \quad . \quad . \quad . \quad (77)$$

We can now understand why H must be expressed in terms of rectangular co-ordinates: if we were to express H otherwise, e.g. in cylindrical or polar co-ordinates, equation (74) would no longer be identical with the equation of propagation (24′) and Heisenberg's method would no longer be justified.

One further remark! Heisenberg's mechanics corresponds to the equation (24′) of the Newtonian wave mechanics and not to the equation (24) of relativistic dynamics.

18. **The meaning of the ν_{ik}'s and of the $q_l{}^{ik}$'s.**—The essential characteristic of Heisenberg's theory is that the ν_{ik}'s are the frequencies of the spectral lines of the atom and that the quantities $q_l{}^{ik}\, e^{2\pi i \nu_{ik} t}$ are related to the intensities of the rays and play the part of electric moment in the classical theories.

Can this be explained by the wave mechanics? So far as the frequencies ν_{ik} are concerned, I had the idea of considering them as frequencies of beats.* The matter might be conceived in the following way, if geometrical optics were valid within the atom. Consider for simplicity the hydrogen atom, and let there be two motions of the electron which are stable in Bohr's sense and which have energies W_i and $W_k < W_i$. In accordance with (27), the waves associated with these two motions would be expressed by

$$\left.\begin{aligned} u_i &= A_i \cos \frac{2\pi}{h}(W_i t - S_i) \\ u_k &= A_k \cos \frac{2\pi}{h}(W_k t - S_k) \end{aligned}\right\}, \quad \ldots \quad (78)$$

and by (33) we must have

$$\int_C dS_i = n_i h, \quad \int_C dS_k = n_k h \quad \ldots \quad (79)$$

along any closed curve C situated in the region where the motion takes place.

Suppose—we shall return to this point—that at a given instant the waves u_i and u_k co-exist in the atom. Their superposition will give rise to beats, the amplitude of which will contain the factor

$$\cos \frac{2\pi}{h}\left[\frac{W_i - W_k}{2} t - \frac{S_i - S_k}{2}\right] = \cos 2\pi \left[\frac{\nu_{ik} t}{2} - \frac{S_i - S_k}{2h}\right]. \quad (80)$$

The sign of the cosine being of no importance, the beats will have the frequency ν_{ik}. Further, on the curve C there are $n_i - n_k$ maxima of the amplitude, and each of them describes the curve with frequency

$$\omega = \frac{\nu_{ik}}{n_i - n_k}, \quad \ldots \quad (81)$$

as follows from (79) and (80).† Thus we see that there appear mechanical frequencies which have a physical meaning and of which the spectral frequencies are harmonics.

On thinking this over, we see that the above conception throws some light on the part played by harmonics in the theory of correspondence. Nevertheless, it is unsatisfactory; to begin with, it rests on the application of the solutions of geometrical optics to the interior of the atom, which is unjustifiable; in the second place, it gives no

* *C. R.*, Vol. **179** (1924), p. 676. See also Schrödinger A, at the end.

† The series of maxima moves round on the curve C and reproduces itself identically with the frequency ν_{ik}.

explanation whatever of the part played by Heisenberg's q_l^{ik}'s defined by the first relation of (73).

In order to interpret the q_l^{ik}'s as components of the electric moment of the atom, Schrödinger * has supposed the undulatory state of the atom to be defined by the superposition of stable waves in the form

$$\Psi(x, y, z, t) = \sum_i a_i u_i(x, y, z)\, e^{2\pi i \nu_i t}, \quad . \; . \; (82)$$

and the electrical density at a point of the atom to be connected with the undulatory state by the relation

$$\rho = \Psi \overline{\Psi}, \quad . \; . \; . \; . \; . \; . \; . \; . \; (83)$$

where $\overline{\Psi}$ is the conjugate imaginary of Ψ. We should thus have:

$$\rho = \Sigma\, a_i a_k u_i u_k \cos 2\pi (\nu_i - \nu_k)\, t, \quad . \; . \; . \; (84)$$

and, from (73), the electric moment of the atom would have for its component in the direction q_l:

$$\mathcal{M}_l = \int \rho q_l \, dv = \Sigma\, a_i a_k q_l^{ik} \cos 2\pi (\nu_i - \nu_k)\, t. \quad . \; (85)$$

The part played by the q_l^{ik}'s would thus be explained. Schrödinger recognizes, however, that this interpretation is open to objections and is only a sketchy one.

Let us return to an important point. It would seem natural to suppose that the internal vibration of the atom in one of its stable states exhibits the frequency ν_i corresponding to that state. Now the two conceptions which have just been expounded have this in common, that they suppose that the waves corresponding to *all* the stable motions are always present in the atom. To explain this apparent paradox, we might perhaps start from the observation that a motion is defined not by a monochromatic wave but by a group of waves. When all the frequencies are stable (e.g. in the case of free motion), the group may be formed of waves with extremely close frequencies; but, if there is a discrete series of rigorously defined stable frequencies, is not the group forced to dissociate itself and to include the whole series of frequencies ν_i which are distantly separated from each other?

19. **Conclusion.**—The interpretation of the quantities q_l^{ik} is bound up with the much more general question of reconciling the wave mechanics with electromagnetism, or rather, of creating a general physics of the field which will include at the same time electromagnetism with suitable modifications and the new mechanics. This physics of the field will have to explain the nature of the

* C, p. 755.

associated waves, justify their equations of propagation, and unravel the deep meaning of the constant h. It will also have to show why the groups of waves with singularities, which constitute matter, are all comprised within a small number of types: the electron, the proton, the quantum of light. . . . We shall then have understood why matter is atomic. We are not there yet, but these questions demand the attention of inquirers from this time forth.

Observations on the Wave Mechanics

BY

LÉON BRILLOUIN

(*Le Journal de Physique et le Radium*, Vol. 7 (1926), p. 353)

Summary.—The fundamental equation of the wave mechanics may be solved by a method of successive approximation; instead of trying to find Schrödinger's ψ function directly, we set

$$\psi = e^{\frac{2\pi i}{h}S}$$

and we obtain an equation of the form

$$H\left(q, \frac{\partial S}{\partial q}\right) - E = -\frac{h}{2\pi i} f\left(\frac{\partial S}{\partial q^k}, \frac{\partial^2 S}{\partial q^k \partial q^l} \cdots\right)$$

for S, where the first member is identical with that of Hamilton's equation,* while the second member involves an expression linear in the partial derivatives of S of the first and second orders. This equation may be solved by approximation by putting

$$S = S_0 - \frac{h}{2\pi i} S_1 + \ldots + \left(-\frac{h}{2\pi i}\right)^n S_n + \ldots,$$

where for S_0 we have the ordinary Hamiltonian equation; the series of equations for the determination of $S_1, S_2, \ldots, S_n$ is then linear. At a given point the function S is many-valued, just like the classical function S_0; its value is determined except for a factor $\Sigma m_k I_k$, where the I_k's are the periods of S. In order that the function ψ may have only a single value, the periodicity of the exponential must compensate for the indeterminateness of S, and this gives the conditions

$$I_k = n_k h \qquad (n_k \text{ a whole number}).$$

By confining ourselves to the first approximation (S_0), we thus obtain the quantized mechanics of Sommerfeld, Schwarzschild, and others. Further approximations add various corrections to the first result (fractional quanta, selection, &c.).

The writer develops and gives a detailed account of this method, which he had already set forth in a note in *Comptes Rendus* (*C. R.*, Vol. **183** (5th July, 1926), p. 24), and which was sketched independently, in a less general form, however, by Wentzel (*Zts. f. Phys.*, Vol. **38** (1926), p. 518).

It is also of interest to seek for the most general type of solvable problems; the writer attempts to do this by generalizing previously obtained results as much as possible.

* [See footnote on p. 12.]

It is proved incidentally that the general equations no longer allow the sign of the time to be reversed, unless the sign of h is changed at the same time; the physical consequence of this will be to replace absorption of energy by emission, and vice versa.

The investigations on the method of solving the equations, which are given in a very general way, enable us, moreover, to specify the form of the matrices and to show by what mechanism the rules of selection are introduced.

1. **Introduction.**—In a quite recent paper, M. L. de Broglie has given an account of the remarkable progress of the wave mechanics,* and has recalled how this theory interprets the existence of stable levels of energy in the atom. In order to study a certain class of motions, such as electronic trajectories, we are led to solve an equation of propagation referring to the associated wave. In the case where relativistic corrections are neglected, i.e. for low velocities, the associated wave has for its wave-surfaces surfaces differing slightly from those given by the Hamilton-Jacobi equation †; ‡ the essential novelty consists in attributing to the wave a frequency ν given by the relation $h\nu = E$. It follows from its law of propagation § that the wave has a high velocity at a distance from the atomic nucleus and a very low velocity near the latter; very strong refraction results, so that the wave goes round the nucleus and returns to interfere with its initial position. There cannot be stability unless the circumstances are such that the refracted wave is in phase with the original wave, and that a state of stationary oscillation is set up. This condition determines a series of particular values of the energy constant E; to each of these values E_k there corresponds a type of stationary waves, that is, a solution $\psi = \psi_k$ of the equation of propagation.

Schrödinger afterwards gave a very ingenious interpretation of the calculations of the matrix mechanics, which throws a new light on that very abstract theory and enables it to be translated into physical terms in the wave mechanics. The method followed is closely connected with the attempts of Born and Wiener, which I mentioned in a recent paper; these writers had already wished to replace non-commutative quantum numbers by functional operators.

The advance made by Schrödinger is very clear in practice. The investigation of a solution of equations involving matrices was very troublesome, since it was necessary to determine all the amplitudes and frequencies at once. The new method enables the problems to be attacked one after the other: finding the levels of energy is a

* *Journal de Physique*, Vol. 7 (1926), p. 321; hereafter denoted *loc. cit.* [p. 55 above].

† [See footnote on p. 12.]

‡ *Loc. cit.*, equations (26′) and (27).

§ *Loc. cit.*, equation (24′) gives:

$$V = E[2m_0(E - F)]^{-\frac{1}{2}},$$

where F is the potential energy.

problem in wave mechanics, and the amplitudes of the terms with which we are concerned are subsequently determined by integrations.

The number of papers that have been published on the wave mechanics has greatly increased in the last few months, a fact which bears witness to the very active interest aroused by these theories.* These ideas had their first origin in some remarks made by my father in 1919;† he pointed out that the quantum conditions may be interpreted as being due to the resonance of a special wave emitted by the electron, and then mentioned that the orders of magnitude involved obliged us to assume that these waves have a much lower velocity close to the nucleus than at a great distance. He added that he had recognized the possibility of defining a continuous medium, which, though analytically unbounded, is asymptotically homogeneous at a great distance, and which *nevertheless possesses proper characteristic periods which are quite distinct from one another.* The wave mechanics has confirmed these views and has given a definite meaning to these hypothetical waves.

2. **The Fundamental Equation.**—The propagation of phase waves appears in its most simple form when the mechanical problem reduces to the motion of a material particle; according as we use relativistic or Newtonian mechanics, we obtain the equations (*loc. cit.*, equation (24)):

$$\nabla^2\psi + \frac{4\pi^2}{h^2c^2}[(W-F)^2 - m_0{}^2c^4]\psi = 0, \quad . \; . \; (1)$$

$$\nabla^2\psi + \frac{8\pi^2}{h^2}m_0(E-F)\psi = 0, \quad . \; . \; . \; . \; (1a)$$

* Louis de Broglie, *Thèse* (1924); *Ann. de Phys.*, Vol. **3** (1925), p. 22; *J. Phys.*, Vol. **7** (1926), p. 1; p. 32; *C. R.*, Vol. **179** (1924), p. 39; p. 676; p. 1039; Vol. **180** (1925), p. 498; Vol. **183** (1926), p. 272.

A. Einstein, *Berlin Ber.* (1925), p. 9.

E. Schrödinger, *Ann. der Phys.*, Vol. **79** (1926), p. 361; p. 489; p. 734; Vol. **80** (1926), p. 437; Vol. **81** (1926), p. 109; *Naturwiss.*, Vol. **14** (1926), p. 664.

E. Fues, *Ann. der Phys.*, Vol. **80** (1926), p. 367.

J. Waller, *Zts. f. Phys.*, Vol. **38** (1926), p. 635.

M. Born, *Zts. f. Phys.*, Vol. **37** (1926), p. 863; Vol. **38** (1926), p. 803.

A. Sommerfeld and A. Unsöld, *Zts. f. Phys.*, Vol. **38** (1926), p. 237.

P. Jordan, *Zts. f. Phys.*, Vol. **37** (1926), p. 376.

L. Brillouin, *C. R.*, Vol. **183** (1926), p. 24; p. 270.

G. Wentzel, *Zts. f. Phys.*, Vol. **38** (1926), p. 518.

Th. de Donder, *C. R.*, Vol. **182** (1926), p. 1380; Vol. **183** (1926), p. 594; *J. Math. Phys. Mass.*, Vol. **5** (1926), p. 251.

Th. de Donder and Fr. H. van den Dungen, *C. R.*, Vol. **183** (1926), p. 22.

O. Klein, *Zts. f. Phys.*, Vol. **37** (1926), p. 895.

Epstein, *Nature*, Vol. **118** (1926), p. 444.

F. London, *Zts. f. Phys.*, Vol. **39** (1926), p. 322.

N. v. Raschevsky, *Zts. f. Phys.*, Vol. **39** (1926), p. 153.

† *C. R.*, Vol. **168** (1919), p. 1318; Vol. **169** (1919), p. 48; Vol. **171** (1920), p. 1000. *J. Phys.*, Vol. **3** (1922), p. 65. *International Congress of Mathematics* at Strasburg (Sept., 1920).

where $F(x, y, z)$ represents the potential energy and W or E the total energy. By taking account of the auxiliary condition

$$h\nu_{\text{rel.}} = W \quad \text{or} \quad h\nu_{\text{cl.}} = E, \qquad (2a)$$

we obtain the values

$$V_{\text{rel.}} = \frac{cW}{\sqrt{(W-F)^2 - m_0^2 c^4}} \quad \text{or} \quad V_{\text{cl.}} = \frac{E}{\sqrt{2m_0(E-F)}} \qquad (3a)$$

for the velocity of propagation.

The first value, $V_{\text{rel.}}$, is extremely great ($V > c$) and of quite a different order of magnitude from $V_{\text{cl.}}$; yet the oscillation problems defined by (1) and (1a) are closely related. This apparent paradox is easily explained. In classical mechanics the energy is undefined to the extent of a constant; we can therefore put

$$E' = E + C; \quad F' = F + C$$

and we obtain a velocity

$$V' = V\left(1 + \frac{C}{E}\right);$$

since, however, the frequency becomes

$$\nu' = \frac{E'}{h} = \nu\left(1 + \frac{C}{E}\right),$$

the wave-length is unaltered ($\lambda' = \lambda$), so that the interference problem remains the same. The relativistic form reduces exactly to the ordinary form in the limit, if we take

$$C = m_0 c^2.$$

Schrödinger has generalized the formulæ relating to a material particle, so as to be in a position to apply them to more complicated problems. The behaviour of a system of r degrees of freedom may be represented by the motion of a point in space of r dimensions. If, then, the expression for the kinetic energy T as a function of the velocities $\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^k, \ldots, \dot{q}^r$ is

$$2T = \sum_{k,l} m_{kl}\, \dot{q}^k\, \dot{q}^l, \qquad (4)$$

we may consider the linear element in the hyperspace of r dimensions as given by

$$ds^2 = \sum_{k,l} m_{kl}\, dq^k\, dq^l, \qquad (5)$$

where the m_{kl}'s represent the fundamental tensor. By means of a familiar transformation we can pass from the covariant components

to the contravariant components m^{kl} of the same tensor. Like the m_{kl}'s, the m^{kl}'s are functions of the various co-ordinates, and by means of these new quantities we can write down the kinetic energy as a function of the momenta p_k:

$$2T = \sum_{k,l} m^{kl} p_k p_l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the hyperspace of the q_k's we obtain an equation of propagation for the wave which is very similar to (1a): it is

$$\text{div grad } \psi + \frac{8\pi^2}{h^2}(E - F)\psi = 0. \quad . \quad . \quad . \quad (7)$$

By working out the vectorial operators, divergence and gradient, in the curved space, we obtain

$$m^{\frac{1}{2}} \sum_{k,l} \frac{\partial}{\partial q^k} \left\{ m^{-\frac{1}{2}} m^{kl} \frac{\partial \psi}{\partial q^l} \right\} + \frac{8\pi^2}{h^2}(E - F)\psi = 0, \quad . \quad (8)$$

where m stands for the determinant of the m^{kl}'s.

All Schrödinger's developments are based on this equation.

This is not a final equation, however, since the quantity E involved in it is connected with ν, the frequency of the function ψ, the latter being supposed to be sinusoidal as regards the time. By using the relation

$$\dot{\psi} = 2\pi i \nu \psi = \frac{2\pi i}{h} E\psi,$$

we can eliminate E, and we obtain the equation:*

$$m^{\frac{1}{2}} \sum_{k,l} \frac{\partial}{\partial q^k} \left\{ m^{-\frac{1}{2}} m^{kl} \frac{\partial \psi}{\partial q^l} \right\} - \frac{4\pi i}{h} \frac{\partial \psi}{\partial t} - \frac{8\pi^2}{h^2} F\psi = 0, \quad . \quad (9)$$

but the fact that imaginaries figure in this equation is somewhat unsatisfactory.

3. **Solution by Successive Approximation.**—Schrödinger has dealt with a great many problems directly, by finding the proper values of the energy E, i.e. the values E_k for which solutions of equation (8) of the form

$$\psi = u_k(q^1, \ldots, q^r) e^{\pm \frac{2\pi i E_k}{h} t} \quad . \quad . \quad . \quad . \quad (10)$$

can be found. The special common feature of the problems treated is this, that they can be solved by separation of variables; the function u has the form of a product of functions $u_i(q^i)$ each of which depends on a single variable. I shall indicate later (§ 6) to what

* Schrödinger introduces a double sign before $\frac{\partial \psi}{\partial t}$ in this equation, which seems unnecessary. (*Ann. der Phys.*, Vol. **81** (1926), p. 109.)

general type these different cases belong. Here I wish to show in a very general way how a solution may be found by successive approximation; this enables a solution to be found in every case where the problem can be solved by ordinary mechanics. This method is interesting first of all in that it specifies what corrections are made in the classical equations by the wave mechanics; besides, it enables a great number of problems to be solved, for the cases of separable variables are much more numerous in classical mechanics than in the wave theory. I sketched this method in a very general form in a note in *Comptes rendus*, and it was independently discovered by Wentzel for some particular cases.*

Let us put $$\psi = e^{\frac{2\pi i}{h}(S+Et)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where S is a function of the co-ordinates q which is to be determined. Simple calculations then enable us to pass from the equations (8 or 9) in ψ to the following one in S:

$$\sum_{k,l} m^{kl} \frac{\partial S}{\partial q^k} \frac{\partial S}{\partial q^l} + 2(F-E) = -\frac{h}{2\pi i} m^{\frac{1}{2}} \sum_{k,l} \frac{\partial}{\partial q^k} \left\{ m^{-\frac{1}{2}} m^{kl} \frac{\partial S}{\partial q^l} \right\}. \quad (12)$$

The terms in the first member of this equation are a reproduction of those of the classical Hamilton-Jacobi equation. In fact, we know that the latter is found by starting from the expression (6) for the kinetic energy as a function of the momenta, and replacing each momentum p_k by the partial derivative $\partial S_0/\partial q_k$ of an unknown function S_0 of the co-ordinates. The condition that the sum of the kinetic and potential energies should be equal to E is next written down, and this gives

$$\sum_{k,l} m^{kl} \frac{\partial S_0}{\partial q^k} \frac{\partial S_0}{\partial q^l} + 2F(q^1, \ldots, q^r) - 2E = 0. \quad . \quad . \quad (13)$$

In the first member we find the very same group of terms as in equation (12); but the wave mechanics adds a second member, and this constitutes the novelty of the theory. This second member is very small, since h is itself very small, so that by making h tend to zero we obtain classical mechanics in the limit. As L. de Broglie remarks, we also arrive at the classical equation when the curvatures of the wave surfaces are of little importance, so that the second derivatives of S are negligible.

In order to solve the general equation (12), we shall attempt to expand S in powers of $\left(\frac{h}{2\pi i}\right)$:

$$S = S_0 - \frac{h}{2\pi i} S_{(1)} + \ldots + \left(\frac{-h}{2\pi i}\right)^n S_{(n)} + \ldots . \quad . \quad (14)$$

* L. Brillouin, *C. R.*, Vol. **183** (1926), p. 24; G. Wentzel, *Zts. f. Phys.*, Vol. **38** (1926), p. 518; L. de Broglie, *loc. cit.*, equations (25–7).

In equation (12) we then collect the terms of the same power of h; the first approximation, involving terms independent of h, is given by the classical equation (13); the second approximation (the terms in h) is given by

$$2\sum_{k,l} m^{kl}\frac{\partial S_0}{\partial q^k}\frac{\partial S_{(1)}}{\partial q^l} = m^{\frac{1}{2}}\sum_{k,l}\frac{\partial}{\partial q^k}\left\{m^{-\frac{1}{2}}\, m^{kl}\frac{\partial S_0}{\partial q^l}\right\}. \quad . \quad (15)$$

At the third stage we have

$$2\sum_{k,l} m^{kl}\frac{\partial S_0}{\partial q^k}\frac{\partial S_{(2)}}{\partial q^l}+\sum_{k,l} m^{kl}\frac{\partial S_{(1)}}{\partial q^k}\frac{\partial S_{(1)}}{\partial q^l} = m^{\frac{1}{2}}\sum_{k,l}\frac{\partial}{\partial q^k}\left\{m^{-\frac{1}{2}}\, m^{kl}\frac{dS_{(1)}}{\partial q^l}\right\}. \quad (16)$$

The general term (that in h^n) takes forms which differ slightly according as n is even or odd:

$$n = 2p;\ 2\sum_{k,l} m^{kl}\frac{\partial S_0}{\partial q^k}\frac{\partial S_{(2p)}}{\partial q^l}+2\sum_{k,l} m^{kl}\frac{\partial S_{(1)}}{\partial q^k}\frac{\partial S_{(2p-1)}}{\partial q^l}+\ldots$$
$$+2\sum_{k,l} m^{kl}\frac{\partial S_{(p-1)}}{\partial q^k}\frac{\partial S_{(p+1)}}{\partial q^l}+\sum_{k,l} m^{kl}\frac{\partial S_{(p)}}{\partial q^k}\frac{\partial S_{(p)}}{\partial q^l}$$
$$= m^{\frac{1}{2}}\sum_{k,l}\frac{\partial}{\partial q^k}\left\{m^{-\frac{1}{2}}\, m^{kl}\frac{\partial S_{(2p-1)}}{\partial q^l}\right\}. \quad . \quad . \quad . \quad . \quad (17)$$

$$n = 2p+1;\ 2\sum_{k,l} m^{kl}\frac{\partial S_0}{\partial q^k}\frac{\partial S_{(2p+1)}}{\partial q^l}+2\sum_{k,l} m^{kl}\frac{\partial S_{(1)}}{\partial q^k}\frac{\partial S_{(2p)}}{\partial q^l}+\ldots$$
$$+2\sum_{k,l} m^{kl}\frac{\partial S_{(p)}}{\partial q^k}\frac{\partial S_{(p+1)}}{\partial q^l} = m^{\frac{1}{2}}\sum_{k,l}\frac{\partial}{\partial q^k}\left\{m^{-\frac{1}{2}}\, m^{kl}\frac{\partial S_{(2p)}}{\partial q^l}\right\}. \quad (17a)$$

These equations are always linear in the partial derivatives of the last unknown function to be determined.

4. **The Proper Values of the Energy and the Quantum Conditions.**—In order to find a satisfactory solution, it is not sufficient to construct a function S by the above method. The proper values of the energy are distinguished by the fact that they enable us to obtain for the function ψ a continuous one-valued determination satisfying the imposed boundary conditions, i.e. vanishing at infinity.

Let us now examine the form exhibited by our solution. The first approximation S_0 is the classical function of action. In all the cases where we know how to form it, it is a many-valued function. At each point $(q^1, \ldots, q^r)$ of the phase extension, it is written as

$$S_0(q^1, \ldots, q^r) + \sum m_l I_{0l},$$

where the m_l's are arbitrary whole numbers, and the I_{0l}'s form a series of r quantities, called the periods of the function S_0. If we start from a given point q and describe any closed circuit, we return

to a new value of the function, corresponding to different values of the whole numbers m_l.

As a consequence of the set of successive approximations, the function S that we are constructing will possess properties similar to those of S_0, with periods I_l slightly differing from the I_{0l}'s; we shall investigate in what cases this multi-valued function S will give us a one-valued expression for ψ. By referring to equation (11), we see that in order to satisfy this condition the periods must be such that

$$I_l = n_l h, \quad \ldots\ldots\ldots \quad (18)$$

where the n_l's are whole numbers.

The indeterminateness of S is then compensated for by the periodicity of the exponential function, since we have

$$\frac{2\pi i}{h} \Sigma m_l I_l = 2\pi i \Sigma m_l n_l.$$

The conditions (18) are the quantum equations which fix the values of the levels of energy.* They appear in a form quite similar to those of the old quantized mechanics of Sommerfeld, Schwarzschild, and others; but instead of relating to the periods I_{0l} of the Hamiltonian function, they apply to the periods of the function defined by the equation (12). The old quantized mechanics here appears as a first approximation.

5. **The Reversibility of the Time.**—Here we shall make a remark which may be of some importance. When the Hamiltonian equation in classical mechanics admits of a solution $+S_0$, it is satisfied by $-S_0$ also.† This directly corresponds to the reversibility of the time; since the momenta are given by the partial derivatives of S_0, changing the sign of this function means changing the sign of the velocities and reversing the sense in which the trajectory is described.

This result essentially depends on the fact that the Hamiltonian equation (13) contains only products of the partial derivatives of S_0 taken two at a time. This state of affairs is not found in our corrected equation (12), however, since the second member is linear.

* In order to obtain the conditions (18), Wentzel proceeds in a different way; his treatment depends on the properties possessed by the function ψ, when a proper solution exists. To me it seemed more straightforward to start from the function S, which can always be obtained from equation (12) for every value of E, and to investigate in what cases this function S gives a suitable form for ψ. It must be noted, however, that the expansion in series (14) does not seem to converge unless the conditions (18) are satisfied.

† I now consider the general case where S_0 is not separable into a sum of functions each involving a single co-ordinate; in the separable case there are as many double signs as there are terms in S_0, and the meaning of these multiple determinations is more complicated.

Thus the reversibility of the time apparently disappears. It can, however, be restored by the following artifice.

If S is a solution of equation (12), $-S$ is a solution of the equation obtained by changing the sign of h. This can also be seen directly for the function ψ in equation (9). The physical result of this is to replace absorption of energy $h\nu$ by emission $(-h\nu)$ and vice versa. This will be the only modification, since the quantum conditions (18) are not affected by this change of sign.*

The double change of sign in S and h enables us to pass from the expansion

$$S = S_0 - \frac{h}{2\pi i} S_{(1)} + \ldots + \left(\frac{-h}{2\pi i}\right)^n S_{(n)} \ldots$$

$$\text{to } S' = -S_0 - \frac{h}{2\pi i} S_{(1)} - \ldots - \left(\frac{+h}{2\pi i}\right)^n S_{(n)} \ldots$$

which may be directly verified from equations (15) to (17).

If we sum the odd and the even terms separately, we obtain real expressions P and $\mathcal{J}$, where

$$P = S_0 + \ldots + \left(\frac{h}{2\pi i}\right)^{2p} S_{(2p)} \text{ and } \mathcal{J} = S_{(1)} + \ldots + \left(\frac{h}{2\pi i}\right)^{2p} S_{(2p+1)}. \quad (19)$$

The two values obtained for ψ are then

$$\psi = e^{-\mathcal{J} + \frac{2\pi i}{h}(P+Et)} \text{ and } \psi' = e^{-\mathcal{J} + \frac{2\pi i}{h}(P-Et)}, \quad . \quad (20)$$

which differ only in the sign of the time.

Thus it seems that the reversibility of the time is obtained without difficulty, at least in the purely mechanical equations; it is not certain, however, that the sign of h can be reversed without further corrections in the equations which express the connexion between the atom and radiation.

6. **The Case of Separable Variables in the Wave Mechanics.**—I considered it worth while to investigate the different types of problems which admit of solution by separation of the variables. This piece of work is indispensable if we afterwards wish to be in a position to establish some general arguments and results, which will apply to all the special examples.

The most simple case is evidently that of complete separability, where F and T appear as sums of terms each of which depends on one variable:

$$F = \sum_k f_k(q^k) \text{ and } 2T = \sum_k m^{kk}(q^k) p_k^2, \quad . \quad . \quad (21)$$

where $f_k(q^k)$ and $m^{kk}(q^k)$ are functions of q_k alone.

* It is in consequence of these remarks that I have continued to suppress the double sign which Schrödinger introduces in front of the term in h in equation (9) and which to me does not seem really justifiable.

It is immediately obvious that we can find a solution for ψ in the form

$$\psi = \psi_1(q^1)\psi_2(q^2)\ldots\psi_k(q^k)\ldots\psi_r(q^r), \quad . \quad . \quad (22)$$

provided that the boundary conditions are independent for each co-ordinate. Then if we divide all the terms of the fundamental equation (8) by ψ, we obtain:*

$$\Sigma\psi_k^{-1}m^{\frac{1}{2}}\frac{\partial}{\partial q^k}\left\{m^{-\frac{1}{2}}m^{kk}\frac{\partial\psi_k}{\partial q^k}\right\} - \frac{8\pi^2}{h^2}F + \frac{8\pi^2}{h^2}E = 0; \quad . \quad (23)$$

moreover,

$$m = m^{11}m^{22}\ldots m^{kk}\ldots m^{rr}.$$

Thus for each variable we deduce an equation

$$\psi_k^{-1}(m^{kk})^{\frac{1}{2}}\frac{\partial}{\partial q^k}\left\{(m^{kk})^{\frac{1}{2}}\frac{\partial\psi_k}{\partial q^k}\right\} - \frac{8\pi^2}{h^2}f_k + \frac{8\pi^2}{h^2}\alpha_k = 0, \quad . \quad (24)$$

which is exactly the form of the general equation (8) for a single degree of freedom. The complete system is formed merely by the juxtaposition of the separate systems, and the total energy is the sum of the partial quantized energies α_k; for the solution of the general equation (23) is given by

$$E = \Sigma\alpha_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

As a rule the matter is not so simple, and there is a connexion between the different degrees of freedom. The cases treated by Schrödinger, Fues, Waller, and others all belong to the following type.†

The terms m^{kl} in the kinetic energy vanish when k is different from l, and the diagonal terms m^{kk} are of the form

$$\left.\begin{aligned} m^{11} &= \mu_1(q^1) \\ m^{22} &= \mu_1(q^1)\mu_2(q^2) \\ m^{kk} &= \mu_1(q^1)\mu_2(q^2)\ldots\mu_k(q^k) \\ m^{rr} &= \mu_1(q^1)\mu_2(q^2)\ldots\mu_k(q^k)\ldots\mu_{r-1}(q^{r-1})\mu_r(q^r) \end{aligned}\right\}. \quad (26)$$

Thus the variables are arranged in a certain order which enables the above classification to be carried out.

The potential energy is assumed to be built up of a sum of terms of similar structure,

$$F = \sum_k \mu_1(q^1)\mu_2(q^2)\ldots\mu_{k-1}(q^{k-1})\chi_k(q^k). \quad . \quad (27)$$

* The quantities m^{kl} vanish, so that all the terms for which $k \neq l$ are destroyed; in those that remain, the functions ψ for which l is not equal to k annul each other in ψ^{-1} and $\frac{\partial\psi}{\partial q^k}$.

† L. Brillouin, *C. R.*, Vol. **183** (1926), p. 270.

From the classical point of view, this problem comes under Stäckel's case of separability. The determinant m of the quantities m^{kl} has the value

$$m = m^{11}\, m^{22} \ldots m^{rr} = \mu_1{}^r\, \mu_2{}^{r-1} \ldots \mu_k{}^{r-k+1} \ldots \mu_{r-1}{}^2\, \mu_r.$$

We can try to find a solution of ψ in the form (22), if the imposed boundary conditions occur independently for each variable; we shall then have the fundamental equation (23). By using the expressions (26) and (27), we shall obtain

$$\sum_k \mu_1 \mu_2 \ldots \mu_{k-1} \left[\psi_k^{-1} \mu_k^{\frac{r-k+1}{2}} \frac{\partial}{\partial q^k} \left\{ \mu_k^{-\frac{r-k+1}{2}+1} \frac{\partial \psi_k}{\partial q^k} \right\} - \frac{8\pi^2}{h^2} \chi_k \right] + \frac{8\pi^2}{h^2} E = 0. \quad (28)$$

The solution is carried out step by step, beginning with the last variable q^r. For the latter, we shall seek for solutions of the equation

$$\psi_r^{-1} \mu_r^{\frac{1}{2}} \frac{\partial}{\partial q^r} \left\{ \mu_r^{\frac{1}{2}} \frac{\partial \psi_r}{\partial q^r} \right\} - \frac{8\pi^2}{h^2} \chi_r + \frac{8\pi^2}{h^2} \alpha_r = 0. \quad . \quad (29)$$

We are concerned only with solutions which are continuous and one-valued and which satisfy the boundary conditions for q^r. This enables us to determine a set of proper values for the constant α_r; after arranging these values in a certain order, we shall write them as $\alpha_r(n_r)$, functions of an integer (n_r).

Let us now see how the problem stands with reference to the $(r-1)$th co-ordinate. For each value of α_r we obtain a different equation, which may be written

$$\psi_{r-1}^{-1} \mu_{r-1} \frac{\partial}{\partial q^{r-1}} \left\{ \frac{\partial \psi_{r-1}}{\partial q^{r-1}} \right\} - \frac{8\pi^2}{h^2} (\chi_{r-1} + \alpha_r \mu_{r-1}) + \frac{8\pi^2}{h^2} \alpha_{r-1} = 0. \quad (30)$$

For each value of the integer n_r we shall now obtain a new set of proper values α_{r-1} depending on another integer n_{r-1}; let the totality of these be represented by $\alpha_{r-1}(n_r, n_{r-1})$, and so on. The kth equation will be

$$\psi_k^{-1} \mu_k^{\frac{r-k+1}{2}} \frac{\partial}{\partial q^k} \left\{ \mu_k^{-\frac{r-k-1}{2}} \frac{\partial \psi_k}{\partial q^k} \right\} - \frac{8\pi^2}{h^2} (\chi_k + \alpha_{k+1} \mu_k) + \frac{8\pi^2}{h^2} \alpha_k = 0, \quad . \quad . \quad (31)$$

and it will give us a set of proper values

$$\alpha_k(n_r, n_{r-1}, \ldots, n_k).$$

The last equation to be solved ($k = 1$) gives us the set of proper values of the energy

$$E = \alpha_1(n_r, n_{r-1}, \ldots, n_2, n_1) \quad . \quad . \quad . \quad . \quad (32)$$

which depends on r integers.

EXAMPLES.

Let us consider the passage from the co-ordinates x, y, z to spherical co-ordinates r, θ, ϕ; the linear element is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$

The kinetic energy of a particle of unit mass is then expressed as a function of the momenta p by

$$2T = p_r^2 + \frac{1}{r^2} p_\vartheta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2,$$

whence we obtain the coefficients of inertia

$$m^{11} = 1,$$
$$m^{22} = \frac{1}{r^2},$$
$$m^{33} = \frac{1}{r^2 \sin^2\theta}.$$

This case does not appear to belong to the type which we have just investigated, as the coefficients m are not of the form stipulated in (26). In reality, however, we can pass from one case to the other by a very simple change of the variables. Moreover, we can immediately formulate the generalization of this example: we should have

$$m^{11} = \lambda_1(q^1),$$
$$m^{22} = \mu_1(q^1)\,\lambda_2(q^2),$$
$$m^{33} = \mu_1(q^1)\,\mu_2(q^2)\,\lambda_3(q^3),$$
$$\ldots\ldots\ldots\ldots$$
$$m^{rr} = \mu_1(q^1)\,\mu_2(q^2) \ldots \mu_{r-1}(q^{r-1})\,\lambda_r(q^r),$$

where each of the functions λ, μ depends on a single co-ordinate. The potential energy would still be given by a formula like (27), and the determinant m would have the value

$$m = \mu_1^{r-1} \mu_2^{r-2} \ldots \mu_k^{r-k} \ldots \mu_{r-1}\, \lambda_1\, \lambda_2 \ldots \lambda_r.$$

The general equation (28) would then be somewhat modified, as follows:

$$\sum_k \mu_1 \mu_2 \dots \mu_{k-1} \left[\psi_k^{-1} \mu_k^{\frac{r-k}{2}} \lambda_k^{\frac{1}{2}} \frac{\partial}{\partial q^k} \left\{ \mu_k^{-\frac{r-k}{2}} \lambda_k^{\frac{1}{2}} \frac{\partial \psi_k}{\partial q^k} \right\} - \frac{8\pi^2}{h^2} \chi_k \right] + \frac{8\pi^2}{h^2} E = 0.$$

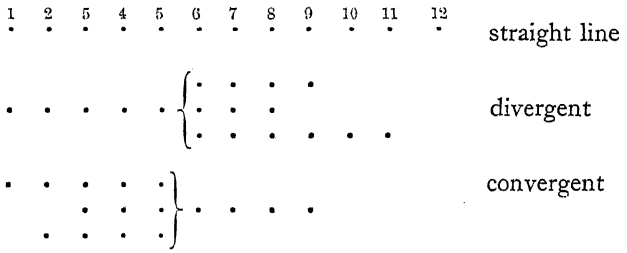
The method of solution will be exactly the same as before; for the separate equation relative to q_k we shall have, instead of (31),

$$\psi_k^{-1} \mu_k^{\frac{r-k}{2}} \lambda_k^{\frac{1}{2}} \frac{\partial}{\partial q^k} \left\{ \mu_k^{-\frac{r-k}{2}} \lambda_k^{\frac{1}{2}} \frac{\partial \psi_k}{\partial q^k} \right\} - \frac{8\pi^2}{h^2} (\chi_k + a_{k+1} \mu_k) + \frac{8\pi^2}{h^2} a_k = 0.$$

7. **Some Generalizations.**—The first case examined (equation (21)) does not at first sight seem to belong to the general type (26); but the quantities $m^{kk}(q^k)$ can always be reduced to the value 1 by a change of co-ordinates, so that we are led to a particular case of (26), where all the μ_k's are also equal to unity.

Various generalizations may be imagined and appear in practice. To begin with, we assumed that the variables were arranged in a unique series, which gave coefficients of inertia (26), and a decomposition of the potential energy in accordance with the scheme (27). I shall call this the straight-line classification.

Starting from this first scheme, however, others may be obtained, which consist of sets of variables connected by branching. Two cases occur, according as the branching is divergent or convergent. The figure below illustrates the principle of these two new methods of arrangement.



In the case of *divergent branching*, we therefore suppose that the variables are arranged as follows:

$$q^1 q^2 \dots q^k \dots q^l \begin{cases} q_0^{l+1} q_0^{l+2} \dots q_0^{l+s_0} \\ q_1^{l+1} q_1^{l+2} \dots q_1^{l+s_1} \\ \dots \dots \dots \dots \\ q_p^{l+1} q_p^{l+2} \dots q_p^{l+s_p} \end{cases}$$

The total number of variables is then

$$r = l + s_0 + s_1 + \dots s_p.$$

The coefficients of inertia m^{kk} are assumed to be of the form (26) for the first set of variables ($k \leqslant l$); for the divergent series, we shall set

$$m_j^{l+1,\,l+1} = \mu_1 \mu_2 \ldots \mu_l \mu_{j,\,l+1}; \quad m_j^{l+\alpha,\,l+\alpha} = \mu_1 \mu_2 \ldots \mu_l \mu_{j,\,l+1} \ldots \mu_{j,\,l+\alpha}. \quad (33)$$

As for the potential energy, it will be given by

$$F = \sum_{k=1}^{k=l} \mu_1 \mu_2 \ldots \mu_{k-1} \chi_k + \sum_{=0}^{j=p} \sum_{\alpha=1}^{\alpha=s_j} \mu_1 \mu_2 \ldots \mu_l \mu_{j,\,l+1} \ldots \mu_{j,\,l+\alpha-1} \chi_{j,\,l+\alpha}; \quad (34)$$

in these formulæ, as in the last section, each of the functions μ, χ depends on a single variable only, that with the same index.

We shall also require to evaluate the determinant m;

$$m = \mu_1^{\,r} \mu_2^{\,r-1} \ldots \mu_l^{\,r-l+1} \mu_{0,\,l+1}^{s_0} \mu_{0,\,l+2}^{s_0-1} \ldots \mu_{0,\,l+s_0} \mu_{1,\,l+1}^{s_1} \mu_{1,\,l+2}^{s_1-1} \ldots \mu_{1,\,l+s_1} \ldots \mu_{p,\,l+1}^{s_p} \ldots \mu_{p,\,l+s_p}.$$

We shall attempt to decompose ψ into a product of factors ψ_k and $\psi_{j,\,l+\alpha}$, each depending on a single co-ordinate.

Then, starting from equation (23), we obtain the general equation:

$$\sum_{k=1}^{k=l} \mu_1 \mu_2 \ldots \mu_{k-1} \left[\psi_k^{-1} \mu_k^{\frac{r-k+1}{2}} \frac{\partial}{\partial q^k} \left\{ \mu_k^{-\frac{r-k-1}{2}} \frac{\partial \psi_k}{\partial q^k} \right\} - \frac{8\pi^2}{h^2} \chi_k \right]$$
$$+ \sum_{j=0}^{j=p} \sum_{\alpha=1}^{\alpha=s_j} \mu_1 \mu_2 \ldots \mu_l \mu_{j,\,l+1} \ldots \mu_{j,\,l+\alpha-1} \left[\psi_{j,\,l+\alpha}^{-1} \mu_{j,\,l+\alpha}^{\frac{s_j-\alpha+1}{2}} \frac{\partial}{\partial q_j^{\,l+\alpha}} \right.$$
$$\left. \left\{ \mu_{j,\,l+\alpha}^{-\frac{s_j-\alpha-1}{2}} \frac{\partial \psi_{j,\,l+\alpha}}{\partial q_j^{\,l+\alpha}} \right\} - \frac{8\pi^2}{h^2} \chi_{j,\,l+\alpha} \right] + \frac{8\pi^2}{h^2} E = 0. \quad (35)$$

In each j series of variables the process of solution is a copy of that used in the last section. We solve step by step, beginning with the last variable $q_{j,\,l+s_j}$; when we come to the beginning of the j series, we shall have determined the set of proper values

$$\alpha_{j,\,l+1}(n_{j,\,s_j}, n_{j,\,s_j-1}, \ldots, n_{j,\,1})$$

depending on s_j integers. The first equation of the principal row will involve the variable q_l and may be written

$$\psi_l^{-1} \mu_l^{\frac{r-l+1}{2}} \frac{\partial}{\partial q^l} \left\{ \mu_l^{-\frac{r-l-1}{2}} \frac{\partial \psi_l}{\partial q^l} \right\}$$
$$- \frac{8\pi^2}{h^2} \left[\chi_l + \mu_l \sum_{j=0}^{j=p} \alpha_{j,\,l+1} \right] + \frac{8\pi^2}{h^2} \alpha_l = 0. \quad . \quad . \quad (36)$$

The series of proper values α_l will then depend on the $s_1 + s_2 + \ldots + s_p$ integers previously introduced, together with a new one. The set

of calculations is completed by the determination of the proper values of $E = \alpha_1$; E will depend on $l + s_1 + s_2 + \ldots + s_p = r$ integers.

As regards the case of *convergent branching*, we have a familiar example of it in parabolic co-ordinates. As is known, this set of variables enables the equations of the Stark effect to be decomposed; after the condition relating to one co-ordinate (the angle ϕ) is solved, two separate similar equations are obtained for the two other co-ordinates. I have not been able to generalize this.

8. **An Examination of the System of Successive Approximations.**—The investigation of the methods for separating the variables has enabled us to find the general form of the equation for one degree of freedom, namely equation (31), which I now write thus:

$$\psi^{-1}\mu^p \frac{\partial}{\partial q}\left\{\mu^{-p+1}\frac{\partial \psi}{\partial q}\right\} - \frac{8\pi^2}{h^2}(\phi - \alpha) = 0, \quad . \quad (37)$$

with the new notation

$$p = \frac{r - k + 1}{2}, \quad \phi = \chi_k + \alpha_{k+1}\mu_k.$$

I have dropped the index k, which is no longer of any use. We shall now compare this equation with that of the old quantum theory by means of the general method described in § 3. To begin with, a difference results from the fact that the coefficient α_{k+1} determined by the previous calculations will not be the same as that given by the old quantum rules;* in particular, in the case of spherical co-ordinates, the solution of the equations for the two angular co-ordinates gives rise to a term $\frac{l(l+1)}{r^2}$ in the equation for r instead of $\frac{l^2}{r^2}$, l being an integer.

Leaving this first modification aside, let us apply the transformation (11). We obtain the equation

$$\mu\left(\frac{\partial S}{\partial q}\right)^2 + 2(\phi - \alpha) = -\frac{h}{2\pi i}\mu^p \frac{\partial}{\partial q}\left\{\mu^{-p+1}\frac{\partial S}{\partial q}\right\} \quad . \quad (38)$$

for S, where μ and ϕ are known functions of q, and α is the constant for which proper values are to be determined.

We shall seek for a solution which is an expansion in powers

* In his applications Wentzel does not clearly recognize this first modification; his calculations therefore do not appear exactly in the form in which I have stated them here and which is, moreover, the general form.

of $-\frac{h}{2\pi i}$, as at (14), and we shall obtain the successive equations:

$$\mu\left(\frac{\partial S_0}{\partial q}\right)^2 + 2(\phi - a) = 0, \quad \ldots \quad (39a)$$

$$2\mu\frac{\partial S_0}{\partial q}\frac{\partial S_1}{\partial q} = \mu^p\frac{\partial}{\partial q}\left\{\mu^{-p+1}\frac{\partial S_0}{\partial q}\right\}, \quad \ldots \quad (39b)$$

$$2\mu\frac{\partial S_0}{\partial q}\frac{\partial S_2}{\partial q} + \mu\left(\frac{\partial S_1}{\partial q}\right)^2 = \mu^p\frac{\partial}{\partial q}\left\{\mu^{-p+1}\frac{\partial S_1}{\partial q}\right\}, \quad \ldots \quad (39c)$$

and so on.

The first equation gives

$$S_0 = \int \pm\sqrt{\frac{2(a-\phi)}{\mu}}\,dq, \quad \ldots \quad (40)$$

and the meaning of this familiar formula is well known. The integral is not real unless the quantity under the square root is positive: the variable q therefore lies between two successive roots q_1, q_2 of this expression. I_0, the period of S_0, is obtained by integration from q_1 to q_2 and back, since the sign before the square root is to be changed at each passage through one of these roots.

The function S_0 is therefore many-valued; its derivative is one-valued on a Riemann surface with a cut from q_1 to q_2, but on the two edges of the cut, which is supposed to be along the x axis, the derivative $\frac{\partial S_0}{\partial q}$ has values which are equal but of opposite sign.

By examining equation (39b) we see that this double value is not carried into the second approximation; changing the sign of $\frac{\partial S_0}{\partial q}$ leaves the derivative $\frac{\partial S_1}{\partial q}$ unaltered. It follows that the function S_1 will have no special features arising from this fact:* it may, however, have one or more poles. In the evaluation of the integral

$$I = \int\frac{\partial S}{\partial q}\,dq \quad \ldots \quad (41)$$

these will give a residue $2\pi iA$, so that we have

$$I = \int\frac{\partial S_0}{\partial q}\,dq + A\,2\pi i\left(-\frac{h}{2\pi i}\right) = I_0 - Ah + \ldots. \quad (41a)$$

* The solution of (39b) is

$$S_1 = \tfrac{1}{2}\log\left\{\mu^{1-p}\sqrt{\frac{2(a-\phi)}{\mu}}\right\};$$

the poles will be the points where μ becomes infinite, together with the points q_1 and q_2, the effects of which annul each other.

According to the old mechanics, we should have set

$$I_0 = nh, \text{ with } n = 0, 1, 2, 3, \ldots \quad (42)$$

If we confine ourselves to the second approximation, represented by S_1, we shall have to write

$$I = I_0 - Ah = nh, \quad (43)$$

which reduces to the previous formula (42), with this difference, however, that the set of possible integers is now

$$n^* = n - A = -A, \quad -A+1, \quad -A+2, \ldots .$$

Since A may be positive or negative, the result of this modification is that n^* may have non-integral values; and if A is an integer, its presence will add some numbers to the previous list, or will, on the other hand, remove some from it. It is by means of this very simple mechanism that the new wave mechanics explains the *exclusion* of some quantized states, and justifies the empirical rules for the introduction of non-integral numbers which were established in the old atomic mechanics.

I shall not dwell on the part played by further approximations, for they are as a rule uninteresting. By writing out the whole series of formulæ (39) it would be easy to see that the derivatives $\frac{\partial S_n}{\partial q}$ have two values on the cut $q_1 q_2$ when n is even, and a single value when n is odd; but these derivatives with double sign in general become infinite at q_1 and q_2, since $\frac{\partial S_0}{\partial q}$ vanishes there. It follows that the two values cannot be interchanged in the integration; thus there will be no unforeseen contributions to the integral I, and only the poles will remain, so that the value of the correction A will be modified. Thus the results will in practice be as stated above.

The constant α is determined by imposing the condition (43), since it occurs in I_0; in fact, the proper values of α are determined in this way.

As at the end of § 5, we can group the set of odd and even terms of approximation under the symbols $\mathcal{J}$ and P; we then have (equation (20))

$$\psi = e^{-\mathcal{J} + \frac{2\pi i}{h}(P + Et)}.$$

S_0, the most important of the even terms, is real between the roots q_1 and q_2 mentioned above, so that the expression $e^{\frac{2\pi i}{h}P}$ will oscillate and ψ will possess a number n of zeros all included in the interval

$q_1 q_2$ to which the motion is limited in classical mechanics. Outside this interval, S_0 becomes imaginary and ψ varies monotonely; now, since the sign of S_0 is at our disposal, we can choose the sign which leaves ψ finite and thus satisfy Schrödinger's boundary conditions.*

9. **The Connexion with the Matrix Mechanics, and the Problem of the Optical Frequencies.**—L. de Broglie has recalled how Schrödinger reconciled the wave mechanics with Heisenberg's theory. Let $u_n(q)$ be the part of ψ which depends only on space co-ordinates: then

$$u_n(q) = e^{\frac{2\pi i}{h} S_n(q)}, \quad \psi_n(q, t) = u_n e^{\frac{2\pi i}{h} E_n t}, \quad . \quad . \quad (44)$$

where q stands for the set of co-ordinates $q^1, \ldots, q^r$, and the u_n's satisfy the relations:†

$$\int \rho u_n u_m \, dq = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } n = m \end{cases}, \quad . \quad . \quad . \quad (45)$$

where the integral is extended throughout the configuration-space.

These conditions for orthogonalism enable us to expand any function whatever in terms of the functions u. Schrödinger obtains the matrices q^k and p^k by expanding the expressions

$$q^k u_n = \sum_m q^k_{nm} u_m \text{ and } \frac{h}{2\pi i} \frac{\partial u_n}{\partial q^k} = \frac{\partial S_n}{\partial q^k} u_n = \sum_m p^k_{nm} u_m. \quad (46)$$

The quantities q^k_{nm} and p^k_{nm} are then the components of the matrices q_k and p_k. We must assume that the expressions (46) are the equivalent of the co-ordinate q^k_n (or of the momentum p^k_n) of the electron, when the atom is in the state of energy E_n; that is, the co-ordinate q^k_n is then given by the sum of the terms of the row n in the matrix q^k, so that in fact we obtain the convention of my previous article.

The components q^k_{nm} and p^k_{nm} may be evaluated as follows:

$$q^k_{nm} = \int \rho q^k u_n u_m \, dq; \quad p^k_{nm} = \int \rho \frac{\partial S_n}{\partial q^k} u_n u_m \, dq. \quad . \quad (47)$$

It may be verified that the coefficients calculated in this way satisfy

* Since this work was published, Kramers has given a very clear statement of these questions, and shown how the apparent half quanta appear in the wave mechanics: *Zts. f. Phys.*, Vol. **39** (1926), p. 828.

† L. de Broglie, *loc. cit.*, equations (39) and (42). Schrödinger always uses the function ρ, which is useless except in the case of non-Cartesian co-ordinates q; its value is given by $\rho = m^{-1}$. Throughout this section the suffix n relates to functions corresponding to a quantized state of energy E_n; care must be taken not to confuse this use with that of some of the preceding sections, where the suffix indicated the order of approximation.

all the relations of the theory of matrices, and must represent the amplitudes of terms of frequency

$$\nu_{nm} = \frac{1}{h}(E_n - E_m). \quad . \quad . \quad . \quad . \quad . \quad (48)$$

We can see immediately how these frequencies ν_{nm} are introduced. The conditions for orthogonalism written down in (45) are not valid unless the functions u are real; if they were imaginary, we should have to write

$$\left.\begin{aligned} &\int \rho u_n u_m^* \, dq = \int \rho u_n^* u_m \, dq = 0, \quad n \neq m \\ &\int \rho u_n u_n^* \, dq = 1 \end{aligned}\right\}, \quad . \quad (49)$$

where the asterisk indicates that the conjugate imaginary is to be taken. In fact it is necessary that the second integral should reduce to a real quantity representing the intensity of the oscillations. This is a familiar procedure in optics; when an oscillatory motion is represented by

$$Ae^{i\omega t},$$

the intensity of the oscillations is A^2 if A is real, whereas, if A is imaginary, the intensity is

$$|A|^2 = AA^*.$$

Instead of the functions u which do not contain the time, we may equally well introduce the functions ψ, and our conditions for orthogonalism will take the form

$$\left.\begin{aligned} &\int \rho \psi_n \psi_m^* \, dq = 0, \quad n \neq m \\ &\int \rho \psi_n \psi_n^* \, dq = 1 \end{aligned}\right\}. \quad . \quad . \quad . \quad (50)$$

Now suppose that we have to decompose a function $\phi(q^k)\psi_n(q, t)$. The two cases mentioned in (46) correspond respectively to

$$\phi = q^k \quad \text{and} \quad \phi = \frac{\partial S_n}{\partial q^k};$$

and we shall have the decomposition

$$\phi\psi_n = \sum_m \phi_{nm}(t)\,\psi_m(q, t), \quad . \quad . \quad . \quad . \quad (51)$$

where the coefficients $\phi_{nm}(t)$ are given by the formulæ

$$\phi_{nm}(t) = \frac{\int \rho\phi\psi_n\psi_m^* \, dq}{\int \rho\psi_m\psi_m^* \, dq} = e^{\frac{2\pi i}{h}(E_n - E_m)t} \frac{\int \rho\phi u_n u_m^* \, dq}{\int \rho u_m u_m^* \, dq} = \phi_{nm} e^{\frac{2\pi i}{h}\nu_{nm}t}. \quad (52)$$

In fact we obtain a term $\phi_{nm}(t)$ which has the frequency ν_{nm} and the amplitude ϕ_{nm}, where ϕ_{nm} is a coefficient of the matrix ϕ.

It appears that we must regard the formulæ (52) which determine the terms $\phi_{nm}(t)$ as more important than the interpretation (51), in which the calculation is represented as the decomposition of a function $\phi\psi_n$ with reference to the proper functions ψ_m. The two methods are mathematically equivalent. From the physical point of view, however, we have the satisfaction of seeing the frequencies ν_{nm} appear in (52); in formula (51), on the other hand, the first term $\phi\psi_n$ has the frequency $\nu_n = \frac{E_n}{h}$, and the second term

$$\Sigma \phi_{nm}(t)\,\psi_m(t) = \Sigma \phi_{nm}\, u_m\, e^{\frac{2\pi i}{h}(E_n - E_m)t}\; e^{\frac{2\pi i}{h}E_m t}$$

also has the frequency ν_n. Moreover, further developments of the theory do not enable us to attribute a physical meaning to these quantities, whereas they enhance the importance of the terms of the matrix $\phi_{nm}(t)$.

10. **The Calculation of the Matrix Coefficients; the Group Velocity as the Velocity of the Moving Body.**—Returning to the calculation of the coefficients of amplitude, e.g. q^k_{nm}, let us suppose that the problem is one of the general type indicated in § 6 involving separable variables, which corresponds to a certain order $q^1, q^2, \ldots, q^k, \ldots, q^r$ of the variables. The proper values are then determined by a set of integers $n_1, n_2, \ldots, n_r$, and we have the following form for the space term of the proper function ψ:

$$u(q;\, n) = u_1(q^1;\, n_1, n_2, \ldots, n_r)\, u_2(q^2;\, n_2, \ldots, n_r) \ldots u_k(q^k;\, n_k, n_{k+1}, \ldots, n_r) \ldots u_r(q^r;\, n_r). \quad (53)$$

The factor $u_k(q_k)$ on the one hand depends on all the integers $n_{k+1}, \ldots, n_r$ which occur in the terms which follow it; on the other hand, it introduces a new integer n_k, which must appear with the same value in all the preceding u's.

Let us then break up the expression

$$\phi(q^k)u(q;\, n) = \sum_m \phi^k_{nm}\, u(q;\, m); \quad . \quad . \quad . \quad . \quad (54)$$

this is an abridged formula, in which q represents the whole of the co-ordinates, and n or m a system of r integers. By examining the expression ϕu, we immediately see that all the terms $u_{k+1}, \ldots, u_r$ are already arranged in a normal series and need not be altered; so we must have

$$m_{k+1} = n_{k+1};\; m_{k+2} = n_{k+2};\; \ldots m_r = n_r.$$

This is the method by which all the *principles of selection* which have been suggested are interpreted in the wave mechanics.

For the k terms we must look for an expansion

$$\phi(q^k)u_k(q^k;\, n_k, n_{k+1}, \ldots, n_r) = \sum_{m_k} a_k(n_k;\, m_k)u_k(q^k;\, m_k, n_{k+1}, \ldots, n_r). \quad (55)$$

But the appearance of an integer m_k different from n_k obliges us to correct all the u's of previous rows; we shall use formulæ like

$$u_{k-1}(q^{k-1};\, n_{k-1}, n_k, \ldots, n_r) = \sum_{m_{k-1}} A_{k-1}(n_{k-1}, n_k;\, m_{k-1}, m_k)u_{k-1}(q^{k-1},\, m_{k-1}, m_k, n_{k+1}, \ldots, n_r)$$

and so on. Finally we shall have coefficients

$$\phi^k(n_1, \ldots, n_r;\, m_1, \ldots, m_k, n_{k+1}, \ldots, n_r) = A_1(n_1, \ldots, n_k;\, m_1, \ldots, m_k)$$

$$A_2(n_2, \ldots, n_k;\, m_2, \ldots, m_k) \ldots A_{k-1}(n_{k-1}, n_k;\, m_{k-1}, m_k)\, a_k(n_k, m_k). \quad (56)$$

If the n's are very large numbers, it is to be expected that in this expansion the only important values of the m's will be those which are very close to the n's. For $\phi u(q, n)$ (equation (54)) we shall therefore have an expansion where there will appear only a group of terms for which m will be nearly equal to n; and this will explain the appearance of the group velocity in the displacement of the moving body, which will justify the statements made by Schrödinger in his second memoir.

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The Universe of Five Dimensions and the Wave Mechanics

BY

LOUIS DE BROGLIE

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Summary.—The object of this paper is to show how remarkably simple an aspect mechanics assumes, in its old form as well as in its new wave form, when the idea of a Universe of five dimensions, which has been brought forward by M. Kaluza, is adopted. The most attractive consequence of this idea is that it makes the notion of force disappear entirely from mechanics, and replaces it by geometrical conceptions, even in the case of the motion of a point charge in an electromagnetic field. Further, thanks to the theory of the Universe of five dimensions, it is possible to put the laws of propagation in the new wave mechanics into a very satisfactory form: this had been mentioned already in an interesting paper by M. O. Klein, but the equation of propagation proposed by him appears to require modification in the way indicated in the present paper.

I.—Introduction

1. **The Notion of Force and General Relativity.**—An outstanding consequence of the principle of equivalence is that it has made the metaphysical notion of force disappear from the theory of gravitation. The motion of a material particle in a gravitational field is defined, according to Einstein, by the simple condition that the line in space-time representing this motion is a geodesic. If the gravitational field vanishes, space-time is Euclidean, and the geodesics are straight lines, as is required by the principle of inertia. If the gravitational field does not vanish, space-time is not Euclidean, and the geodesics are no longer straight lines, so that the space-trajectories of material points are curves.

The success of this beautiful interpretation of the gravitational field makes it very tempting to expunge the whole notion of force from Physics and replace it by ideas drawn from metrical geometry, but there is a difficulty that has to be taken into account. In the present state of our knowledge, it appears that all the forces which we know

from experience to exist are reducible to two types only: gravitational forces and electromagnetic forces. As we have said, the first type of force can be entirely reduced to the idea of curvature of space-time, but the same cannot be said of the second. In fact, the trajectory of a point charge in a region where there is an electromagnetic field but where the gravitational field is imperceptible, is not rectilinear, although space-time is Euclidean: so the world-line of the charge is not a geodesic. Further, particles having the same mass but different charges will have different trajectories, so that no class of curves in space-time defined by an *intrinsic* property can be identified with the possible world-lines of point charges, as these lines depend on the nature of the moving body, through the ratio e/m_0 of the charge to the proper mass.

In order to perfect Einstein's work and reduce electromagnetic force to geometrical quantities, M. Kaluza * has developed a bold but very elegant theory: the theory of relativity in five dimensions. M. O. Klein † has shown that this five-dimensional relativity enables the equations of the new wave mechanics to be written in a remarkably symmetrical form. As all these conceptions are rather unfamiliar to physicists, and as, moreover, I prefer to write some of the equations in M. Klein's paper in a different form, I think it advisable to summarize the dynamical side of the question at this point.

II.—The Point of View of the Non-undulatory Mechanics

2. **Motion of a Point in a Gravitational Field.**—Let us investigate the motion of a particle of proper mass m_0 in a gravitational field where the metrical relation

$$ds^2 = g_{ik}\, dx^i\, dx^k \qquad \ldots\ldots \quad (1)$$

holds, with the usual convention ‡ as to summation of the indices.

Changing the sign usually chosen, we shall call the curvilinear integral

$$A(M) = \int_O^M m_0 c\, ds, \qquad \ldots\ldots \quad (2)$$

taken along the world-line from an origin O to the point M, the Hamiltonian action of the point. Hamilton's principle then shows that the world-lines are geodesics in space-time.

* Kaluza, *Sitzungsber. d. Berl. Akad.* (1921), p. 966.

† O. Klein, *Zts. f. Phys.*, Vol. **36** (1926), p. 895.

‡ [A. S. Eddington, *Mathematical Theory of Relativity*, § 22.]

The equations of motion are very easily obtained by writing

$$\delta \int_0^M m_0 c\, ds = \delta \int_0^M m_0 c\, g_{ik} \frac{dx^i}{ds}\, dx^k = \delta \int_0^M m_0 c\, g_{ik}\, u^i\, u^k\, ds = 0, \quad (3)$$

and noticing that, *since the extremals sought are geodesics*, $s(M)$ does not vary (to the first order) when the path of integration is varied. The classic Euler-Lagrange equations corresponding to (3) are therefore *:

$$\frac{d}{ds}(m_0 c\, g_{ik}\, u^k) = \tfrac{1}{2}\, m_0 c \frac{\partial g_{\mu\nu}}{\partial x^i}\, u^\mu\, u^\nu. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These are the equations of motion, and it is easy to verify that they lead to the Newtonian theory of gravitation as a first approximation. This point is so familiar nowadays that we need not dwell on it further.

3. **Motion of a Point Charge in an Electromagnetic Field.**—For simplicity, suppose that the gravitational field is negligible, and take rectangular axes. From the relativity point of view, the electromagnetic field is defined by a vector ϕ in space-time whose components are given as a function of the scalar potential ψ and of the vector potential $\mathbf{a}$ by the relations:

$$\phi_4 = \phi^4 = \frac{\psi}{c}, \quad \phi_1 = -\phi^1 = -\frac{a_x}{c},$$

$$\phi_2 = -\phi^2 = -\frac{a_y}{c}, \quad \phi_3 = -\phi^3 = -\frac{a_z}{c}. \quad (5)$$

We know that the electric and magnetic fields are defined by the formulæ:

$$\mathbf{h} = -\operatorname{grad}\psi - \frac{1}{c}\frac{\partial \mathbf{a}}{\partial t}; \quad \mathbf{H} = \operatorname{curl}\mathbf{a}. \quad . \quad . \quad (6)$$

The force acting on the charge e is, according to Lorentz †:

$$\mathbf{f} = e\left\{\mathbf{h} + \frac{1}{c}[\mathbf{v} \times \mathbf{H}]\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7]$$

The Hamiltonian action of the moving body must here be written as

$$A(M) = \int_0^M (m_0 c + e\phi_s)\, ds, \quad . \quad . \quad . \quad . \quad (8)$$

where ϕ_s is the component of ϕ along the tangent to the world-line. As ϕ_s in general depends on the x_i's, the world-line is no longer a geodesic and depends on the ratio e/m_0.

* [T. Levi-Civita, *Absolute Differential Calculus*, p. 131; A. S. Eddington, *loc. cit.*, § 28.] † [O. W. Richardson, *Electron Theory of Matter*, Chapter IX.]

The extremals of the problem of Hamiltonian variation being no longer geodesics, we write, in order to have an integral with non-variable limits,

$$\delta \int_{t_0}^{t_M} L\, dt = 0, \qquad \ldots \ldots \quad (9)$$

where $L = m_0 c^2 \sqrt{1-\beta^2} + e\,(c\phi_4 + v_x\phi_x + v_y\phi_y + v_z\phi_z)$,

v_x, v_y, v_z being the components of the velocity in the usual sense. The Lagrangian equations are then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i} \qquad (i = 1, 2, 3), \quad \ldots \quad (10)$$

and it is easy to verify that they have the vector form:

$$\frac{d}{dt}\left(\frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}}\right) = e\,\mathbf{f}. \qquad \ldots \ldots \quad (11)$$

These are the dynamical equations of the electron where the notion of force holds good.

4. **Five-dimensional Relativity.**—Let us imagine with M. Kaluza that, in order to represent the series of events in the Universe, it is necessary to employ a manifold of five dimensions; that is, a fifth dimension, corresponding to a fifth variable x^0, is to be added to space-time. The variations of this fifth variable are quite beyond our senses, so that two points of the Universe corresponding to the same values of the four variables of space-time but to different values of the variable x^0 are indistinguishable. We are, as it were, shut up in our space-time manifold of four dimensions, and we perceive only the projections on this space-time of points in the Universe of five dimensions.

Under these conditions, the element of a world-line will be given by the formula:

$$d\sigma^2 = \gamma_{00}\,(dx^0)^2 + 2\sum_1^4 \gamma_{0i}\, dx^0\, dx^i + \sum_1^4 \sum_1^4 \gamma_{ik}\, dx^i\, dx^k, \qquad (12)$$

where we shall suppose all the γ's independent of the co-ordinate x^0.

In passing from one system of axes of reference to another in motion with respect to the first, we can perform all sorts of change of variables in space-time, but these changes of variables cannot affect the variable x^0. Thus it seems logical to say that all changes of variables which are *humanly* possible are of the form

$$x_i' = f_i(x_1, x_2, x_3, x_4) \qquad (i = 1, 2, 3, 4). \quad . \quad (13)$$

The quantity

$$\gamma_{00}\,(dx^0)^2 + \sum_1^4 \gamma_{0i}\,dx^0\,dx^i$$

is therefore an invariant *for us*, and we see immediately that the same is true for

$$ds^2 = \left(\gamma_{ik} - \frac{\gamma_{0i}\,\gamma_{0k}}{\gamma_{00}}\right) dx^i\,dx^k. \quad . \quad . \quad . \quad . \quad (14)$$

In formula (14) and those which follow the summation is to be taken only for the values 1, 2, 3, 4, corresponding to space-time.

Let us then put

$$d\theta = \sqrt{\gamma_{00}}\,dx^0 + \frac{\gamma_{0i}}{\sqrt{\gamma_{00}}}\,dx^i. \quad . \quad . \quad . \quad . \quad (15)$$

This is an invariant for us, and we can evidently write

$$d\sigma^2 = ds^2 + d\theta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In order to arrive at an interpretation of electromagnetic phenomena, it is necessary to introduce the potentials. Since the γ_{0i}'s are transformed like vectors in space-time, we are led to set

$$\gamma_{0i} = \alpha\gamma_{00}\,\phi_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where the ϕ_i's are defined by (5), and α is a constant of homogeneity.

The equation $d\theta = 0$ expresses the fact that the infinitesimal displacement under consideration is normal to the direction x^0, that is, to the intersection of the surfaces $x^1 =$ constant, $x^2 =$ constant, $x^3 =$ constant, $x^4 =$ constant, an intersection which is not in general normal to our space-time $x^0 =$ constant. When γ_{0i} is defined as in (17), the equation $d\theta = 0$ is integrable only when the electric and magnetic fields vanish; thus, in general, there does not exist a manifold of four dimensions everywhere normal to the direction x^0. It is natural to suppose, however, that the element of every world-line satisfying the condition $d\theta = 0$ (that is, everywhere normal to the fifth dimension) has a measure completely determined by the phenomena of gravitation, and consequently that it is given by the expression $g_{ik}\,dx^i\,dx^k$ of Einstein's theory. Using (16) and (14), we are led to set

$$\gamma_{ik} = g_{ik} + \frac{\gamma_{0i}\,\gamma_{0k}}{\gamma_{00}} = g_{ik} + \gamma_{00}\,\alpha^2\,\phi_i\,\phi_k. \quad . \quad . \quad (18)$$

It is interesting to picture the matter to oneself geometrically with the help of a diagram (fig. 1). In this figure a co-ordinate line x^0 is drawn. At a point P on this line, a small portion of the four-dimensional manifold $x^0 =$ constant which passes through P

is represented by an element of the plane π, which is oblique to the direction x^0. $\overline{PQ}$ is an element of the world-line, of length $d\sigma$; $\overline{PS}$, its orthogonal projection on the direction x^0 (the *covariant* component of $d\sigma$ along x^0); $\overline{PR}$, its projection in a direction perpendicular to x^0. Formula (16) shows at once that we have

$$\overline{PS} = d\theta; \quad \overline{PR} = ds.$$

We shall see later (formula (26)) that $d\theta/ds$, the tangent of the angle QPR, is proportional to the ratio e/m_0 for the material particle for which PQ is the element of the world-line. Hence it follows that the world-line of every such particle makes the same angle with the direction x^0 at every point; if the electric charge vanishes, the angle is a right angle.

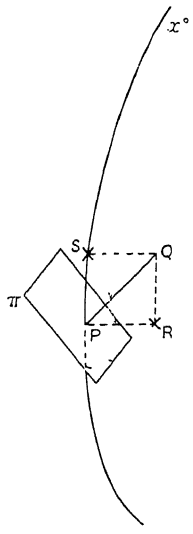

Fig. 1.—$\overline{PQ} = dr, \overline{PS} = d\vartheta, \overline{PR} = ds$

It is therefore clear that the co-ordinate lines x^0 possess in some sense an absolute character, and that our space-time may be considered as some four-dimensional section or other of the five-dimensional Universe, the intersections of that section with the lines in the direction of x^0 alone having an absolute meaning. This can be explained if we suppose with Klein that the lines x^0 are closed and that their total length is less than anything we can measure; the Universe would be, so to speak, thread-like in the direction normal to the fifth dimension.

By means of a principle of variation, Einstein's law of gravitation and at the same time Maxwell's equations can be deduced from the conception of the Universe of five dimensions.* I shall not dwell on this side of the question here and shall confine myself to noting the following consequence of this reasoning; the product $\gamma_{00}\alpha^2$ is connected with G, the usual constant of gravitation, by the relation:

$$\gamma_{00}\,\alpha^2 = -\frac{16\pi G}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

5. The World-line of every Material Particle is a Geodesic.—If the existence of a fifth dimension of the Universe is assumed, the following principle may be asserted: "*In the five-dimensional Universe, the world-line of every material particle is a geodesic.*" We shall verify this by showing that the principle does

* O. Klein, *loc. cit.*, pp. 897–9.

lead to the equations of Einstein's dynamics, provided a hypothesis is made which determines the geometrical meaning of electric charge.

We shall define a world-velocity by five components,

$$u^i = \frac{dx^i}{d\sigma} \qquad (i = 0, 1, 2, 3, 4), \qquad . \quad . \quad . \quad (20)$$

and to define the geodesics we shall write

$$\delta \int_O^M \tfrac{1}{2}\, d\sigma = \delta \int_O^M \tfrac{1}{2} \left[\gamma_{00}\,(u^0)^2 + 2\,\gamma_{0i}\,u^0\,u^i + \gamma_{ik}\,u^i\,u^k\right] d\sigma = 0, \quad (21)$$

where O and M are two fixed points of the Universe. As the extremals are geodesics, we have to vary an integral of the form $\int L d\sigma$ taken between fixed limits (see § 2), and we can write down Lagrange's equations:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial u^i}\right) = \frac{\partial L}{\partial x^i} \qquad (i = 0, 1, 2, 3, 4). \quad . \quad . \quad (22)$$

Bearing (15) in mind, we therefore obtain, to begin with,

$$\gamma_{00}\,u^0 + \gamma_{0i}\,u^i = \sqrt{\gamma_{00}}\,\frac{d\theta}{d\sigma} = \text{constant} = p_0. \qquad . \quad (23)$$

Then for the variables with indices 1, 2, 3, 4, we have:

$$\frac{d}{d\sigma}\,(g_{ik}\,u^k + \alpha\,p_0\,\phi_i) = \tfrac{1}{2}\,u^\mu\,u^\nu\,\frac{\partial g_{\mu\nu}}{\partial x^i} + \alpha\,p_0\,u^\mu\,\frac{\partial \phi_\mu}{\partial x^i}. \quad . \quad (24)$$

If the electromagnetic field vanishes, we have only to take s as independent variable instead of σ to recover equation (4) of Einstein's theory. If the gravitational field vanishes, the g_{ik}'s are constants, and we have:

$$\frac{d}{d\sigma}(g_{ik}\,u^k) = \alpha\,p_0\,u^\mu\left(\frac{\partial \phi_\mu}{\partial x^i} - \frac{\partial \phi_i}{\partial x^\mu}\right) = \alpha\sqrt{\gamma_{00}}\,\frac{d\theta}{d\sigma}\left(\frac{\partial \phi_\mu}{\partial x^i} - \frac{\partial \phi_i}{\partial x^\mu}\right) u^\mu. \quad (25)$$

Suppose that the inclination of the world-line of a moving body of mass m_0 and charge e to the direction x^0 is defined by the fundamental relation:

$$\frac{d\theta}{ds} = \frac{e}{\alpha\sqrt{\gamma_{00}}\,m_0 c}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Equation (25) can then be written:

$$-\frac{d}{ds}\left(\frac{dx^i}{ds}\right) = \frac{e}{m_0 c}\,\frac{dx^\mu}{ds}\left(\frac{\partial \phi_\mu}{\partial x^i} - \frac{\partial \phi_i}{\partial x^\mu}\right) \qquad (i = 1, 2, 3), \quad (27)$$

and, by using the definitions (5), we readily revert to the equations (11) in rectangular co-ordinates.

Thus, with the geometrical meanings which we have assigned to the potentials and to the ratio e/m_0, the five-dimensional world-lines of material particles are always geodesics. *The notion of force has been entirely banished from mechanics.*

Since the constants α and γ_{00} cannot depend on the properties of the moving body under consideration, equation (26) leads us to set

$$m_0 c = I\frac{ds}{d\sigma}, \quad \frac{e}{\alpha\sqrt{\gamma_{00}}} = I\frac{d\theta}{d\sigma}, \quad . \quad . \quad . \quad . \quad (28)$$

where I is an invariant * such that

$$I^2 = m_0{}^2 c^2 + \frac{e^2}{\alpha^2 \gamma_{00}}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

in virtue of (16).

In order to obtain the Hamiltonian action of formula (2) in the case of zero charge, we shall write:

$$A(M) = \int_0^M I\, d\sigma = \int_0^M I\left(\frac{d\theta}{d\sigma}d\theta + \frac{ds}{d\sigma}ds\right)$$
$$= \int_0^M \left(\frac{e}{\alpha\sqrt{\gamma_{00}}}d\theta + m_0 c\, ds\right) = \frac{e}{\alpha}x_0 + f(x_1, x_2, x_3, x_4). \quad (30)$$

III.—The Point of View of the Wave Mechanics

6. The Wave Mechanics and Four-dimensional Relativity.—I have explained the principles of the wave mechanics in a previous article.† The essential idea of this new theory is that of considering matter as an undulatory phenomenon of nature, represented by a function which satisfies certain equations of propagation. When the conditions are realized which enable the solutions of the equation of propagation to be investigated by means of the methods of geometrical optics, each unit of matter may be compared to a group of monochromatic waves with frequencies comprised within a very small interval $\nu - \delta\nu$, $\nu + \delta\nu$. Superposition of the waves in the group gives rise to a singular point of phase agreement, which is displaced along one of the *rays* of the central wave of frequency ν, and this singular point, which is the analytical representation of the material particle, moves according to the laws of the old dynamics.

* The quantities (28) are the components, in and normal to the direction of x^0, of the vector I along the tangent to the world-line. This vector is clearly the five-dimensional generalization of the world-impulse.

† *Journal de Physique*, Vol. 7 (Nov., 1926), p. 321. [P. 55.]

When the approximations of geometrical optics are no longer valid, the solutions of the equations of propagation require rigorous investigation: it is here that the new mechanics is more general than the old, and the fertility of this generalization, to which M. Schrödinger's fine results have already borne witness, is still far from being entirely exhausted.

In the case of motion in the absence of any field, we found the equation of propagation (in rectangular Galilean co-ordinates) to be:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{4\pi^2 m_0{}^2 c^2}{h^2} u, \quad \ldots \quad (31)$$

where m_0 is a characteristic constant of the moving body (its proper mass). Using (four-dimensional) relativity notation, we may write (31) in the form

$$g^{ik}\frac{\partial^2 u}{\partial x^i \partial x^k} = -\frac{4\pi^2}{h^2} m_0{}^2 c^2 u, \quad \ldots \quad (32)$$

where, as usual,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

Now suppose there is a gravitational field, but no electromagnetic field. We cannot keep to the form (32), because the expressions $\frac{\partial^2 u}{\partial x^i \partial x^k}$ are not components of a tensor. Introducing the covariant derivative * of the gradient of u, we are led to replace equation (32) by the invariant equation

$$g^{ik}\left[\frac{\partial^2 u}{\partial x^i \partial x^k} - \left\{\frac{ik}{r}\right\}\frac{\partial u}{\partial x^r}\right] = -\frac{4\pi^2}{h^2} m_0{}^2 c^2 u, \quad (33)$$

with the well-known notation †:

$$\left\{\frac{ik}{r}\right\} = \tfrac{1}{2} g^{r\mu}\left(\frac{\partial g_{\mu i}}{\partial x^k} + \frac{\partial g_{\mu k}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^\mu}\right). \quad \ldots \quad (34)$$

Equation (33) will be the equation of propagation of material waves in the gravitational field. Let us verify that, to the degree of approximation of geometrical optics, we obtain the Einsteinian dynamics of the gravitational field. In fact, the central wave of the group corresponding to a unit of matter can then be expressed by

$$u = Ce^{\frac{2\pi i}{h}\phi}, \quad \ldots \quad (35)$$

where C is a constant.

Substitute in (33) the value of u given by (35); as we are using the approximations of geometrical optics, retain in the first member

* [T. Levi-Civita, *loc. cit.*, pp. 85, 147; A. S. Eddington, *loc. cit.*, § 29.]

† [T. Levi-Civita, *loc. cit.*, p. 110; A. S. Eddington, *loc. cit.*, § 27.]

only the terms of the second degree in the first differential coefficients of u. Equation (33) becomes:

$$g^{ik}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^k}=m_0{}^2c^2, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

whence we obtain:

$$\phi=m_0c\int_0^M u_i\,dx^i=\int_0^M m_0c\,ds. \quad . \quad . \quad . \quad (37)$$

ϕ is thus identical with the Hamiltonian action defined by (2), and the rays of the central wave (35) are geodesics in space-time. As the trajectory of a material particle is one of these rays, we come back to the Einsteinian dynamics of the material particle in a gravitational field.

7. **The Wave Mechanics and the Five-dimensional Universe.**—Equation (33) is valid when electromagnetic phenomena are absent. In order to take account of them by introducing Kaluza's ideas, it is sufficient to generalize equation (33) by assuming that the periodic phenomenon, matter, in the five-dimensional Universe, satisfies the equation:

$$\gamma^{ik}\left[\frac{\partial^2u}{\partial x^i\partial x^k}-\left\{\begin{matrix}ik\\r\end{matrix}\right\}\frac{\partial u}{\partial x^r}\right]=-\frac{4\pi^2}{h^2}I^2u, \quad . \quad . \quad (38)$$

where I is the invariant defined by (29), an invariant which is naturally introduced in place of m_0c in the case of electrified points.*

If the methods of geometrical optics are valid, it can be shown as in the preceding paragraph that the central wave of the group defining a material particle is expressed by

$$Ce^{\frac{2\pi i}{h}A}=f(x,y,z,t)\,e^{\frac{2\pi i}{h}.\frac{ex_0}{\alpha}},$$

where A is the action of formula (30). It follows that the five-dimensional world-line of every material particle is a geodesic, and consequently we obtain the Einsteinian dynamics of gravitation and the dynamics of the electron, to the degree of approximation of geometrical optics.

Without any longer supposing geometrical optics to apply, let us find out what equation (38) becomes in the absence of a gravitational field. The g_{ik}'s then assume, in rectangular co-ordinates, their Galilean values, and the relations (17) and (18) give the γ_{ik}'s. Thence the γ^{ik}'s are easily deduced. Since, from (19), the product $\gamma_{00}\alpha^2$ is of the order of 10^{-27} c.g.s. units, terms of that order may be assumed negligible compared to unity. We can no longer suppose that u is of the form

$$Ce^{\frac{2\pi i}{h}A},$$

* In equation (38) the indices of course vary from 0 to 4.

but we still consider u to be the product of a function of x, y, z, t by $\sin\frac{2\pi}{h}\cdot\frac{ex_0}{a}$. This being so, an easy calculation enables us to write (38), to the degree of approximation considered, in the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + \left[\frac{1}{\gamma_{00}} + \frac{\alpha^2}{c^2}(\psi^2 - a^2)\right]\frac{\partial^2 u}{\partial x_0^2} - 2\sum_{x,y,z}\alpha\frac{a_x}{c}\frac{\partial^2 u}{\partial x^0\,\partial x} - \frac{2\alpha\psi}{c^2}\frac{\partial^2 u}{\partial x^0\,\partial t} = -\frac{4\pi^2 I^2}{h^2}u, \quad (39)$$

or, bearing in mind the form assumed for u,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u - \frac{4\pi i}{h}\sum_{x,y,z}\frac{ea_x}{c}\frac{\partial u}{\partial x} - \frac{4\pi i}{h}\frac{e\psi}{c^2}\frac{\partial u}{\partial t} + \frac{4\pi^2}{h^2}\left[m_0{}^2 c^2 - \frac{e^2}{c^2}(\psi^2 - a^2)\right]u = 0. \quad (40)$$

This is equation (59) of my previously mentioned article* in the *Journal de Physique*. After giving this equation, I added: "It is, however, to be observed that (59) contains imaginary terms, and this perhaps raises some objections from the physical point of view." It is obvious that the anomaly disappears here, since (40) is only a degenerate form of (39).

8. **Conclusion.**—Equation (38), which, in virtue of (29) and (19), can further be written in the very remarkable form

$$\gamma^{ik}\left[\frac{\partial^2 u}{\partial x^i\,\partial x^k} - \left\{\begin{matrix} ik \\ r \end{matrix}\right\}\frac{\partial u}{\partial x^r}\right] + \frac{4\pi^2 c^2}{h^2}\left[m_0{}^2 - \frac{e^2}{16\pi G}\right]u = 0, \quad (41)$$

appears to be the general equation of the wave mechanics of a material particle. In order to get to the bottom of the problem of matter and its atomic structure, it will no doubt be necessary to study the question systematically from the view-point of the five-dimensional Universe, which seems more fertile than M. Weyl's point of view. If we succeed in interpreting the way in which the constants e, m_0, c, h, and G enter into equation (41), we shall be very close to understanding some of the most perplexing secrets of Nature.

* [P. 70.]

The Wave Mechanics and the Atomic Structure of Matter and of Radiation

BY

LOUIS DE BROGLIE

(*Le Journal de Physique et le Radium*, Vol. 8 (1927), p. 225)

Summary.—Schrödinger and most of the writers who have dealt with wave mechanics have sought to represent dynamical phenomena by the propagation of waves of continuous amplitude of the type familiar in optics. At first sight it is difficult to understand how this point of view can be reconciled with the atomic structure of matter and of radiation, which is now scarcely contested any longer. The object of the present paper is to show that continuous solutions in reality provide only a certain statistical view of dynamical phenomena, the exact description of which probably requires the consideration of waves which admit of singularities. In particular, this kind of conception enables a clear meaning to be given to the equation which Schrödinger proposes for the dynamics of systems.

I.—Introduction*

The object of the wave mechanics is to create a synthesis embracing both the dynamics of a material particle and the theory of waves as conceived by Fresnel. On the one hand, the effect of this synthesis must be to introduce the idea of points of concentration of radiant energy into optics, an idea which at present seems to be required by the recent results of experimental physics; on the other hand, it must introduce the conceptions of the theory of waves into our picture of material particles, in order to account for the occurrence of quanta in mechanics, and for intra-atomic phenomena.

The new mechanics defines the possible motions of material particles by means of equations of propagation whose form depends on potential functions. In order to represent the motion of a material particle (electron, proton, or photon), can we adopt a continuous solution without singularities of the equation of propagation,

* This paper is the development of two notes which appeared in *Comptes Rendus*, Vol. **183** (1926), p. 447, and Vol. **184** (1927), p. 273.

analogous to those used in Fresnel's optics? Such solutions evidently do not account for the atomic structure of matter at all: it seems to me physically preferable to attempt to represent each material particle by a corresponding solution of the equation of propagation, whose amplitude involves a point-singularity which is the analytical representation of the existence of the material particle. Nevertheless, in optics the use of continuous solutions has for a century enabled physicists to foretell phenomena very precisely, and, besides, Schrödinger has just successfully used continuous solutions to represent the stationary states of micromechanics. These verifications lead us to ask whether there is not a connexion between continuous solutions and solutions with singularities of the equations of propagation, a connexion which might be broadly expressed as follows: the continuous solutions would give a statistical representation of the displacement of the singularities corresponding to real solutions, and they would consequently enable the " probability of presence " of a singularity in a given volume of the space where the motion takes place to be calculated.

This is the idea which I am going to try to develop and formulate precisely here, making a frank statement of the postulates which I assume and which it would be desirable to justify. My conception approaches the one brilliantly upheld by Born in this, that it leads us to consider the continuous solutions as giving the probabilities of presence, but it differs from it in an essential point. For Born, indeed, there is nothing but probabilities: the deterministic view of individual phenomena should be abandoned, since the probability of statistical phenomena is alone determinate. In the way of looking at the question adopted here, on the contrary, the material particle is an essential reality, and its motion is completely determined as that of a singularity in the amplitude of a wave which is propagated. Only, just as in the old mechanics, the motion of the particle depends on the initial conditions, and if these initial conditions are ignored (at least to an extent which will be specified), we may speak of the probability that the material particle is found at a given instant in a given element of volume of the space; this is the probability that would be obtained by considering continuous waves. It would in this way be possible to retain the atomic structure of matter and of radiation, as well as the determinism of individual phenomena, while at the same time attributing to the continuous solutions the statistical meaning which Born and implicitly Schrödinger have recognized in them.

II.—Continuous Waves and the Dynamics of a Material Particle

A. *The Case of No Field*

1. **The Equation of Propagation and its Solutions.**—Let us consider a particle of proper mass m_0 situated outside all fields in a region where no obstacle exists. This particle is in uniform motion in a straight line (or at rest) with respect to a Galilean system of axes, and we learn from the wave mechanics that this motion must be compared with the propagation of a wave and represented by a solution of the following equation,*

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{4\pi^2 \nu_0^2}{c^2} u, \quad \ldots \quad (1)$$

where

$$\nu_0 = \frac{m_0 c^2}{h}. \quad \ldots \quad (2)$$

By reasoning similar to that in my thesis [cf. § 13, p. 68 above], we are led to seek solutions of (1) of the form

$$u(x, y, z, t) = f(x, y, z, t) \cos \frac{2\pi\nu_0}{\sqrt{1-\beta^2}}\left[t - \frac{\beta z}{c} + \tau\right], \quad (3)$$

or, putting

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}} = \frac{W}{h}, \quad V = \frac{c}{\beta}, \quad \ldots \quad (4)$$

$$u(x, y, z, t) = f(x, y, z, t) \cos 2\pi\nu\left[t - \frac{z}{V} + \tau\right]. \quad (5)$$

In the first formula of (4), W stands for the total energy of the moving body, including its internal energy $m_0 c^2$.

If the argument of the cosine is written in the form $\frac{2\pi}{h}\phi(x, y, z, t)$, the function ϕ is none other than Hamilton's action.

Again, it follows from (4) that the velocity of the material point is equal to the velocity of a group of plane homogeneous waves of the form

$$A \cos 2\pi\nu\left[t - \frac{z}{V} + \tau\right],$$

that is, we have

$$\frac{1}{v} = \frac{\partial(\nu/V)}{\partial\nu}. \quad \ldots \quad (6)$$

This remarkable fact makes one think that the material particle may be compared to a group of monochromatic waves. This idea

* See *Journal de Physique*, Series VI, Vol. 7 (Nov., 1926), pp. 321–37, equation 49. I shall refer to this paper by the letters J. P. [P. 55.]

of mine has been taken up by Schrödinger and has led him to consider the material particle as a " parcel of waves ".* From the didactic point of view it is very convenient to use this representation, but it is uncertain whether it corresponds to reality, for, as I shall show, equation (6) can be obtained without appealing to the notion of a group of waves.

To make this point quite definite, we must ask what form the function $f(x, y, z, t)$ of equations (3) and (5) can take. It seems physically possible that this function possesses a singularity where the material particle is, and that the series of values of f is transferred as a whole parallel to the direction of motion with the velocity v. Thus we should write the function as $f(x, y, z - vt)$, and if we then substitute the function (5) in its complex form in equation (1), we obtain

$$vV = c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

by equating imaginary terms to zero.

Now this relation, which is equivalent to the second equation of (4), leads to (6), and consequently (6) has been obtained without supposing that the solution of equation (1) can be represented by a group of homogeneous waves of close frequencies. This way of interpreting formula (6) is at bottom much more in accordance with the considerations that I used in my thesis than the conception of a group of waves.

By equating real terms to zero after substituting in (1) from (5), we obtain a second equation:

$$\Box f \equiv \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We know that this equation is invariant for the Lorentz transformation: the function f must therefore satisfy Laplace's equation for a system of axes fixed in the moving body, if we suppose that the wave phenomenon is stationary in this system. The most simple hypothesis then consists in assuming that in this proper system the material particle possesses spherical symmetry: f is then a function of the radius r_0 alone, and we must have

$$u(x_0, y_0, z_0, t_0) = \frac{C}{r_0}\cos 2\pi\nu_0 (t_0 + \tau_0). \quad . \quad . \quad (9)$$

If the material particle, instead of being spherically symmetrical, had cylindrical symmetry about the axis of x_0, we could take for our solution the function

$$u(x_0, y_0, z_0, t_0) = \frac{Cx_0}{r_0^3}\cos 2\pi\nu_0 (t_0 + \tau_0) \quad . \quad . \quad (10)$$

instead of (9).

* Ger. *Wellenpaket.*

The function u being thus found for the proper system, to get its expression in another Galilean system it is sufficient to carry out a Lorentz transformation. If we return, for example, to the system where the body moves along the z axis with velocity v, solution (9) becomes

$$u(x, y, z, t) = \frac{C}{\sqrt{x^2 + y^2 + \frac{(z - vt)^2}{1 - \beta^2}}} \cos 2\pi\nu \left[t - \frac{z}{V} + \tau \right]. \quad (11)$$

The above solution, and analogous solutions such as (10), correspond to the old mechanics in this sense, that the phase is proportional to the Hamiltonian action. It is curious to find that there are other solutions of (1) of the same form whose counterparts in the old mechanics do not exist. For example, let us take the proper system considered above and seek for a solution of the form $f \sin 2\pi\nu_0' t$, where $\nu_0' \neq \nu_0$: we shall have to satisfy the relation:

$$\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial y_0^2} + \frac{\partial^2 f}{\partial z_0^2} = \frac{4\pi^2}{c^2} (\nu_0^2 - \nu_0'^2) f. \quad . \quad . \quad (12)$$

We then find, for "solutions with spherical symmetry", the functions:

$$\left.\begin{aligned} f(r_0) &= \frac{C}{r_0} \cos 2\pi \left[\frac{\sqrt{\nu_0'^2 - \nu_0^2}}{c} r_0 + C' \right] \text{ if } \nu_0' > \nu_0 \\ f(r_0) &= \frac{1}{r_0} \left[C e^{\frac{2\pi}{c}\sqrt{\nu_0^2 - \nu_0'^2}\, r_0} + C' e^{-\frac{2\pi}{c}\sqrt{\nu_0^2 - \nu_0'^2}\, r_0} \right] \text{ if } \nu_0 > \nu_0' \end{aligned}\right\}, \quad (13)$$

and we shall thence deduce a solution for any Galilean system by means of a Lorentz transformation.* If the line of motion is still taken as the z axis, this solution will be of the form:

$$u(x, y, z, t) = f(x, y, z - vt) \cos \frac{2\pi\nu_0'}{\sqrt{1 - \beta^2}} \left[t - \frac{\beta z}{c} \right]. \quad (14)$$

We may say that motions unknown in the old dynamics are thus obtained, in which the moving body, instead of having its normal proper mass m_0, would have an abnormal proper mass $h\nu_0'/c^2$. The deviation from the normal mechanical state presented here is characterized by a non-zero value of $\Box f$.

We are thus led to the following general standpoint: the wave mechanics of the free material particle is given by equation (1), *some* of the solutions of which correspond to the old dynamics: but there exist other solutions, of which formulæ (13) are examples. These

* Of course, in the second equation (13), we must take $C = 0$ if the wave phenomenon has to vanish at infinity.

other solutions indicate possible states of motion unforeseen by the old theories; the content of equation (1) is therefore much richer than that of the differential equations of the old dynamics.

2. **The Representation of a Swarm of Particles by a Continuous Wave.**—Let us consider a swarm of material particles of the same kind, which are under no external or mutual forces and which all have the same velocity v in the same direction Oz. If the wave phenomenon is given at every point by the normal form (5), the phenomenon as a whole will be represented by the function

$$U(x, y, z, t) = \sum_i f_i(x, y, z - vt) \cos 2\pi\nu \left[t - \frac{vz}{c^2} + \tau_i \right]. \quad (15)$$

We shall make the simplifying assumption (to which, as we shall see farther on, we must not attach too great importance) that the quantities τ_i are equal. The particles are then in the same phase, and we may write:

$$U = \left[\sum_i f_i(x, y, z - vt) \right] \cos 2\pi\nu \left(t - \frac{vz}{c^2} \right). \quad . \quad (16)$$

The amplitude given by the term in square brackets permits of a great many singularities moving with velocity v parallel to Oz.

Now, equation (1) also admits of the continuous solution *

$$\Psi(x, y, z, t) = a \cos 2\pi\nu \left(t - \frac{vz}{c^2} \right), \quad . \quad . \quad . \quad (17)$$

where a is a constant.

We shall say that this continuous solution corresponds to the solution with singularities given by (5). We shall call the number of corpuscles per unit volume the density of the swarm, and we shall suppose that this density has everywhere the constant value ρ. As a, the constant of the continuous solution (17), may be chosen arbitrarily, we shall put

$$\rho = Ka^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where K is a constant given in advance. We shall see that the continuous solution will give the phase distribution in the swarm of particles by means of its trigonometrical factor, while the square of its amplitude will be a measure of the density of the swarm.

In a fluid where the density at the point (x, y, z) is $\rho(x, y, z)$, the probability that a molecule taken at random is in an element of volume dv surrounding the point considered is $\rho(x, y, z)dv$. This

* Here we shall always denote the continuous solutions of the equations of propagation by Ψ: our functions Ψ are thus identical with those that Schrödinger designates by this letter.

observation will enable us to present what follows in a different form. Let us consider a *single* material particle in uniform motion in a straight line: suppose that the magnitude and direction of its velocity are known, but that its position is unknown. Then the product a^2dv will measure the probability that the particle is within the element dv at any instant. Thus we see that the previously assumed condition of equality of the τ_i's is not essential, since the swarm of particles considered above may now be looked upon as consisting of the series of possible positions of a single particle.

B. *The Case of Constant Fields*

3. The Propagation of a Material Particle in a Constant Field.—We shall first consider the case of a constant field defined by a potential function $F(x, y, z)$. The wave mechanics then assumes the following equation* of propagation, which the wave in its complex form must satisfy:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{4\pi i}{h}\frac{F(x,y,z)}{c^2}\frac{\partial u}{\partial t} - \frac{4\pi^2}{h^2}\left[m_0^2 c^2 - \frac{F^2}{c^2}\right]u = 0. \quad (19)$$

We shall imagine that the moving body at first moves in a region R_0 of space where the function F vanishes, and then penetrates into the region R where the field under consideration exists. In the region R_0, equation (19) reduces to (1) and the material particle is then represented, according to us, by the function

$$u(x, y, z, t) = f(x, y, z, t)\cos\frac{2\pi}{h}\phi. \quad . \quad . \quad (20)$$

The function f involves a moving singularity and the function ϕ is the Hamiltonian action of the old mechanics. To obtain the wave representation of the material point where the field of force exists, solution (20) must be extended into the region R. Let us see what relations f and ϕ must satisfy there: in order to do this, write (20) in its complex form, substitute in (19), and separate real from imaginary parts. We obtain the two equations:

$$\frac{1}{f}\Box f = \frac{4\pi^2}{h^2}\left[\Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{2F}{c^2}\frac{\partial\phi}{\partial t} + \left(m_0^2 c^2 - \frac{F^2}{c^2}\right)\right], \quad (21_1)$$

$$\Sigma\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial x} - \frac{1}{c^2}\frac{\partial f}{\partial t}\frac{\partial\phi}{\partial t} + \tfrac{1}{2}f\Box\phi + \frac{F}{c^2}\frac{\partial f}{\partial t} = 0. \quad . \quad (21_2)$$

The field being constant, the region R is analogous to a refracting medium of constant properties, and when the wave penetrates into

* [Cf. pp. 70, 111 above.]

it, it will remain monochromatic with the frequency $\nu = \frac{W}{h}$ that it has in R_0: in the new mechanics this expresses the fact that in a constant field the energy remains constant. Thus we have:

$$\phi(x, y, z, t) = Wt - \phi_1(x, y, z); \quad \frac{\partial \phi}{\partial t} = W = h\nu; \quad \Box\phi = \nabla^2\phi. \quad (22)$$

Equation (21_1) may then be written:

$$\frac{1}{f}\Box f = \frac{4\pi^2}{h^2}\left[\Sigma\left(\frac{\partial \phi_1}{\partial x}\right)^2 - \frac{1}{c^2}(W - F)^2 + m_0{}^2 c^2\right]. \quad . \quad (23)$$

If the first member were negligible, this relation would be identical with Jacobi's equation in the relativistic dynamics of constant fields and ϕ_1 would be Jacobi's function. Thus here the deviation from the old mechanics appears to be bound up with the non-vanishing of $\Box f$.

According to the old approximation, the velocity of the particle passing through a point M is in the direction of the vector grad ϕ_1 at that point. We shall *assume* that the same is true when f and ϕ are determined rigorously.

According to our ideas, the material particle is a singularity of the function f where the latter becomes infinite in the inverse ratio of a certain power of the distance. If we denote by n a variable reckoned in any direction which passes through M, the position of the moving body at the instant t, we therefore have

$$\left[\frac{f}{\frac{\partial f}{\partial n}}\right]_{M, t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Take n along the normal at M to the surface $\phi_1(x, y, z) =$ constant and apply equation (21_2), remembering (22) and (24), and we obtain:

$$\left[\begin{array}{c} \frac{\partial f}{\partial t} \\ -\frac{\partial f}{\partial n} \end{array}\right]_{M, t} = \frac{c^2 \operatorname{grad}\phi_1}{W - F}. \quad . \quad . \quad . \quad . \quad (25)$$

By the assumption we made about the direction of the velocity, it is therefore given by:

$$v_M = \frac{c^2 \operatorname{grad}\phi_1}{W - F}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

To the degree of approximation of the old mechanics, ϕ_1 is identical with Jacobi's function, and the relation (26) is then the

one which connects momentum and velocity in Einstein's dynamics. The object of the above reasoning is to make it appear probable that relation (26) is strictly true in the new mechanics.

We shall specify farther on (Part III) what is meant, in wave mechanics, by the Newtonian approximation, and we shall see that to this degree of approximation the denominator of (26) can be replaced by m_0c^2, so that we then simply have:

$$\mathrm{v}_M = \frac{1}{m_0}\,\mathrm{grad}\,\phi_1. \qquad . \quad . \quad . \quad . \quad . \quad (26')$$

4. **The Propagation of a Swarm of Particles in a Constant Field.**—Now let there be identical material particles without mutual interaction, which at the beginning of their motion are traversing the region R_0 with the same velocity in the same direction. If these particles are in the same phase, in the sense previously explained, the swarm can be represented by the function (16) in R_0. We shall assume that the extension of this function into the region R still has a unique phase factor, or, in other words, that the function $\phi_1(x, y, z)$ of the preceding paragraph is the same for all the particles of the swarm. These particles will then have velocities defined by the relation (26), and their motion is comparable to the steady motion of the molecules in a fluid, since the velocity of a particle at the time of passing through a point depends only on the position of the point and not on the time of passage. Since it is sufficient to employ the relation (26'), the function ϕ_1 plays the part of velocity potential.

According to our ideas, the velocities are always along the tangents to the orthogonal curves of the family of surfaces $\phi_1 = \text{constant}$: these curves are therefore stream-lines, and they form tubes within which the particles are displaced. As these tubes are not of constant section in the region R, ρ, the density of the fluid, varies from point to point while remaining constant at each point, since the motion is steady. In this case the hydrodynamical equation of continuity gives us a relation that the function $\rho(x, y, z)$ must satisfy:

$$\mathrm{div}\,\rho\mathbf{v} = 0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Using (26), we may write (27) in the form

$$\frac{\partial}{\partial n}\left[\log\left(\frac{\rho}{W-F}\right)\right] = -\frac{\nabla^2\phi_1}{\mathrm{grad}\,\phi_1}. \qquad . \quad . \quad . \quad (28)$$

As in the case of uniform motion, we shall seek to represent the swarm of particles by a continuous wave. The swarm can be represented in the region R_0 by the continuous wave (17), where the density and amplitude are connected by the relation (18). When the continuous wave (17) penetrates into the region R, where

propagation is according to (19), it will be represented therein by a function of the form

$$\Psi(x, y, z, t) = a(x, y, z) \cos \frac{2\pi}{h} \phi'(x, y, z, t)$$
$$= a(x, y, z) \cos 2\pi \left[\nu t - \frac{1}{h} \phi_1'(x, y, z) \right]. \quad . \quad (29)$$

The region R_0 is analogous to a homogeneous refracting medium, the region R to a non-homogeneous refracting medium: thus the determination of the functions a and ϕ_1' reduces to the solution of a problem of classical optics.

If we write the solution (29) in its complex form and substitute in equation (19), we shall obtain two relations by separating real from imaginary parts:

$$\frac{1}{a} \nabla^2 a = \frac{4\pi^2}{h^2} \left[\Sigma \left(\frac{\partial \phi'}{\partial x} \right)^2 - \frac{1}{c^2} \left(\frac{\partial \phi'}{\partial t} \right)^2 + \frac{2F}{c^2} \frac{\partial \phi'}{\partial t} + \left(m_0^2 c^2 - \frac{F^2}{c^2} \right) \right]. \quad (30_1)$$

$$\Sigma \frac{\partial a}{\partial x} \frac{\partial \phi'}{\partial x} + \frac{1}{2} a \nabla^2 \phi' = 0. \quad . \quad . \quad . \quad . \quad (30_2)$$

The form of ϕ' enables us to write

$$\frac{1}{a} \nabla^2 a = \frac{4\pi^2}{h^2} \left[\Sigma \left(\frac{\partial \phi_1'}{\partial x} \right)^2 - \frac{1}{c^2} (h\nu - F)^2 + m_0^2 c^2 \right] \quad (31)$$

instead of (30_1).

If the first member is negligible, the equation of geometrical optics corresponding to the equation of propagation (19) is obtained. Comparing equations (23) and (31), we see that if their first members are negligible we obtain the old mechanics from the one and geometrical optics from the other. Thus the functions ϕ_1 and ϕ_1' are identical; they coincide with Jacobi's function.

We shall now make the essential assumption that ϕ_1 and ϕ_1' are still identical when the first members of equations (23) and (31) are no longer negligible. This obviously requires us to have

$$\frac{1}{a} \nabla^2 a = \frac{1}{f} \sqcup f. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

We shall call this postulate the " principle of the double solution ", because it implies the existence of two sinusoidal solutions of (19) with the same phase factor, the one involving a point-singularity and the other, on the contrary, a continuous amplitude. This principle is, of course, a provisional one, in the sense that it must be capable of confirmation, or the reverse, by rigorous reasoning: but it is forcibly suggested by the necessity of reconciling the atomic

structure of matter and of light with the successes of classical optics and of Schrödinger's theory.

The function ϕ_1' being thus identified with ϕ_1, the relation (30_2) will become

$$\frac{2}{a}\frac{\partial a}{\partial n} = \frac{\partial}{\partial n}[\log a^2] = -\frac{\nabla^2\phi_1}{\text{grad}\,\phi_1}, \quad . \quad . \quad . \quad (33)$$

whence, by comparison with (28), we conclude that the quantity $\frac{\rho}{a^2(W-F)}$ remains constant along a stream-line. Since relation (18) holds in the region R_0 where F vanishes, in R we must have

$$\rho(x, y, z) = Ka^2(x, y, z)\left[1 - \frac{F(x, y, z)}{W}\right]. \quad . \quad (34)$$

If the function F is given, it is obvious that the determination of the continuous wave (29) must give the density of the swarm at each point.

When it is legitimate to neglect the potential energy as compared with the total energy (the Newtonian approximation), the approximate formula may be written

$$\rho(x, y, z) = Ka^2(x, y, z). \quad . \quad . \quad . \quad . \quad (34')$$

We can evidently consider the preceding in a different way by supposing that the particles of the swarm are just *repetitions of the same particle*. We in fact suppose the initial velocity in the region R_0 given in magnitude and direction: if we know nothing more, that is to say if all the initial *positions* of the moving body are equally probable, a motion will correspond to each assumption as to the initial position, and, superposing all these possibilities in imagination, we shall obtain the equivalent of the motion of an infinitely dense swarm of identical particles. Then the probability that a particle should be actually present at a given instant in an element of volume dv surrounding the point (x, y, z) of the region R is evidently proportional to $\rho(x, y, z)\,dv$: this probability is therefore given, as a function of the quantities belonging to the continuous wave, by the formulæ (34), (34'). Moreover, the form of the trajectories is likewise determined from a knowledge of the continuous wave, since the trajectories are orthogonal to the surfaces of equal phase.

As an example, let us consider a swarm of electrified particles, with the same velocity in the same direction, which have just passed through the neighbourhood of a fixed charged centre. According to the distance of its initial rectilinear trajectory from this centre, each particle will be more or less deviated, and the old mechanics enables us to calculate the proportion of the particles which will be

deviated in a given direction, when the density of the incident swarm is assumed uniform: this is the calculation made by Sir E. Rutherford in order to find the scattering of α-rays by matter. The new mechanics adopts a different point of view and considers that the space surrounding the centre refracts the waves of the incident electrons. If the ideas proposed above are exact, it should be possible to obtain the statistical law of scattering as follows: consider a plane continuous wave falling upon a refracting sphere, the refractive index of which is a function of the distance from the centre varying according to a suitable law, and calculate the intensities scattered in different directions: these intensities ought to give the relative proportions of electrons scattered in these directions. This indeed appears to result from an interesting calculation by G. Wentzel,* who has in this way obtained Rutherford's law as a first approximation.

C. *The Case of Variable Fields*

5. **The Equation of Propagation.**—The equation of propagation corresponding to the motion of an electric charge e in an electromagnetic field defined by a scalar potential $\mathcal{V}(x, y, z, t)$ and a vector potential $\mathbf{A}(x, y, z, t)$ is: †

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{4\pi i}{h}\frac{e\mathcal{V}}{c^2}\frac{\partial u}{\partial t} + \frac{4\pi i}{h}\Sigma\frac{e}{c}A_x\frac{\partial u}{\partial x} - \frac{4\pi^2}{h^2}\left[m_0{}^2 c^2 - \frac{e^2}{c^2}(\mathcal{V}^2 - A^2)\right]u = 0. \quad (35)$$

In principle, the terms containing the vector potential must always be put in, for, according to the relation of Lorentz, the vector potential cannot vanish if the scalar potential is variable. If, nevertheless, the influence of these terms is negligible, we can content ourselves by writing:

$$\Box u + \frac{4\pi i}{h}\frac{F}{c^2}\frac{\partial u}{\partial t} - \frac{4\pi^2}{h^2}\left[m_0{}^2 c^2 - \frac{F^2}{c^2}\right]u = 0, \quad . \quad (35')$$

where $e\mathcal{V} = F(x, y, z, t)$.

We shall still suppose that the moving body under consideration begins its motion in a region of space R_0 where the potentials are zero and then penetrates into a region R where the variable field exists. We shall again attempt to extend the solution of type (5), which holds in R_0, by a solution of (35) of the form

$$u(x, y, z, t) = f(x, y, z, t)\cos\frac{2\pi}{h}\phi(x, y, z, t), \quad . \quad (36)$$

* *Zts. f. Phys.*, Vol. **40** (1926), p. 590. I had suggested this result in my book *Ondes et Mouvements*, p. 84.

† J. P. equation 59. [Pp. 70, 111.]

where f exhibits a moving singularity. Substituting in (35), we still obtain two equations:

$$\frac{1}{f}\Box f = \frac{4\pi^2}{h^2}\left[\Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + 2\frac{e\mathcal{V}}{c^2}\frac{\partial\phi}{\partial t} + 2\frac{e}{c}\Sigma A_x\frac{\partial\phi}{\partial x} + m_0{}^2c^2 - \frac{e^2}{c^2}(\mathcal{V}^2 - A^2)\right], \quad (37_1)$$

$$\Sigma\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial x} - \frac{1}{c^2}\frac{\partial f}{\partial t}\frac{\partial\phi}{\partial t} + \tfrac{1}{2}f\Box\phi + \frac{e\mathcal{V}}{c^2}\frac{\partial f}{\partial t} + \frac{e}{c}\Sigma A_x\frac{\partial f}{\partial x} = 0. \quad (37_2)$$

If the first member of (37_1) were negligible, $\phi(x, y, z, t)$ would be the Jacobi function of the relativistic dynamics of variable fields. The deviation from the old mechanics is thus still bound up with the non-vanishing of $\Box f$.

According to the approximation of the old theories, the velocity of the particle passing through $M(x, y, z)$ at the instant t is in the direction of the vector momentum, which is defined by the relation

$$\mathbf{g} = -\left[\operatorname{grad}\phi + \frac{e}{c}\mathbf{A}\right]. \quad \ldots\ldots \quad (38)$$

As before, we shall assume that the same is true in the case of the rigorous solutions of the equation of propagation.

It is clear that relation (24) is still valid here. We shall make use of it by choosing n as the variable reckoned in the direction of the vector $\mathbf{g}$ at the instant t and the point M. Equation (37_2) then becomes:

$$\left[-\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial n}}\right]_{M,t} = \frac{c^2 g}{\frac{\partial\phi}{\partial t} - e\mathcal{V}}, \quad \ldots\ldots \quad (39)$$

and the assumption as to the direction of the velocity gives

$$\mathbf{v}(M, t) = \frac{c^2\,\mathbf{g}}{\frac{\partial\phi}{\partial t} - e\mathcal{V}}, \quad \ldots\ldots \quad (40)$$

a relation of which (26) is evidently a particular case. To the approximation of the old mechanics, equation (40) is that connecting momentum and velocity.

Here again, if the energy of motion is small compared with the internal energy m_0c^2, we have

$$\mathbf{v}(M, t) = \frac{1}{m_0}\,\mathbf{g}. \quad \ldots\ldots \quad (40')$$

6. **The Propagation of a Swarm of Particles.**—It is quite clear that we may apply the considerations of paragraph 4 to the case of variable fields. We shall again consider a swarm of identical particles, "in phase" and without interaction, which, at the beginning of their motion, are traversing the region R_0 in the same direction with the same velocity. This swarm, the density of which is supposed uniform in R_0, will be represented in R_0 by the function (16); when extended into the region R, this solution would exhibit a single phase if the approximations of the old mechanics were valid, the phase being then given by Jacobi's function. We shall assume that in the strict solution the phase is still unique, that is to say, the swarm can be represented in R by the function

$$U(x, y, z, t) = [\Sigma_i f_i(x, y, z, t)] \cos \frac{2\pi}{h} \phi(x, y, z, t), \qquad (41)$$

where the function f_i exhibits a moving singularity.

The velocities are given by the formula (40), but, as the motion is of course not steady, the equation of continuity must be written:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \operatorname{div}\left[\frac{\rho c^2 \mathbf{g}}{\frac{\partial \phi}{\partial t} - e\mathcal{U}}\right] = 0. \qquad (42)$$

Let us introduce the notation

$$\rho' = \rho \cdot \frac{1}{\frac{\partial \phi}{\partial t} - e\mathcal{U}}. \qquad (43)$$

Using (38) and the Lorentz relation between the potentials, we readily obtain the equation:

$$g \frac{\partial(\log \rho')}{\partial n} + \frac{1}{c^2}\left[\frac{\partial \phi}{\partial t} - e\mathcal{U}\right] \frac{\partial(\log \rho')}{\partial t} = \Box \phi. \qquad (44)$$

As before, we shall seek to represent the propagation of a swarm by that of a continuous wave of classical type. In R_0 this wave will be of the form (17), where the density and the amplitude are connected by the relation (18). The region R plays the part of a refracting medium, the refractive index of which varies with the time at every point, and, when the continuous wave penetrates into R, it will take the form:

$$\Psi(x, y, z, t) = a(x, y, z, t) \cos \frac{2\pi}{h} \phi'(x, y, z, t). \qquad (45)$$

(45) differs from (29) in that a depends on the time and ϕ' is no longer linear in t. Ψ must of course satisfy the equation

of propagation (35), and this as usual leads to two relations:

$$\frac{1}{a}\Box a=\frac{4\pi^2}{h^2}\left[\Sigma\left(\frac{\partial\phi'}{\partial x}\right)^2-\frac{1}{c^2}\left(\frac{\partial\phi'}{\partial t}\right)^2+2\frac{e\mathcal{V}}{c^2}\frac{\partial\phi'}{\partial t}+\frac{2e}{c}\Sigma A_x\frac{\partial\phi'}{\partial x}+m_0^2c^2-\frac{e^2}{c^2}(\mathcal{V}^2-A^2)\right]; \quad (46_1)$$

$$\Sigma\frac{\partial a}{\partial x}\frac{\partial\phi'}{\partial x}-\frac{1}{c^2}\frac{\partial a}{\partial t}\frac{\partial\phi'}{\partial t}+\tfrac{1}{2}a\Box\phi'+\frac{e\mathcal{V}}{c^2}\frac{\partial a}{\partial t}+\frac{e}{c}\Sigma A_x\frac{\partial a}{\partial x}=0. \quad (46_2)$$

We shall again introduce the principle of the double solution, assuming that the function ϕ' is identical with the function ϕ, that is to say, that to solution (41), the amplitude of which involves singularities, there must correspond a solution with continuous amplitude and the same phase factor. This leads, moreover, by comparison of (37_1) and (46_1), to the equality:

$$\frac{1}{a}\Box a=\frac{1}{f}\Box f. \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

This being so, equation (46_2) gives:

$$g\frac{\partial(\log a^2)}{\partial x}+\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}-e\mathcal{V}\right)\frac{\partial(\log a^2)}{\partial t}=\Box\phi. \quad . \quad (48)$$

Comparing this with (44), we see that if we follow the motion of the same particles of the swarm the quotient $\frac{\rho'}{a}$ remains constant. Since the relation (18) holds in R_0, we have, at every point of R and at every instant:

$$\rho(x,y,z,t)=\frac{K}{W_0}a^2(x,y,z,t)\left[\frac{\partial\phi}{\partial t}-e\mathcal{V}\right]=K'a^2\left[\frac{\partial\phi}{\partial t}-e\mathcal{V}\right]. \quad (49)$$

If the kinetic energy is small compared with the internal energy m_0c^2, the density of the swarm can still be considered proportional to the square of the amplitude of Ψ in this case.

The swarm can of course be regarded as formed by the whole series of possible positions of the same material particle, of which only the velocity in R_0 in magnitude and direction is known. The probability that the moving body is at the instant t within a volume dv surrounding the point (x, y, z) is $\rho(x, y, z, t)dv$, and this is determined by equation (49) as a function of the quantities belonging to the continuous wave.

7. **The Current Vector in the Swarm of Electrified Particles.**—In accordance with a well-known procedure, we can at every point of the swarm and at every instant define a current

four-vector, the components of which are, putting $ict = x_4$,

$$s_1 \quad \rho c \frac{v_x}{c}, \quad s_2 \quad \rho c \frac{v_y}{c}, \quad s_3 \quad \rho c \frac{v_z}{c}, \quad s_4 \quad i\rho c. \quad . \quad (50)$$

Using (38), (40), and (49) we readily obtain

$$s_1 \quad K' e a^2 c \left[\frac{\partial \phi}{\partial x} + \frac{e}{c} A_x \right], \ldots, s_4 \quad ic K' a^2 \left[\frac{\partial \phi}{\partial t} - e\mathcal{V} \right]. \quad (51)$$

Let us introduce the four-vector $\mathbf{P}$ with components

$$P_1 \quad A_x, \ P_2 \quad A_y, \ P_3 \quad A_z, \ P_4 \quad i\mathcal{V}. \quad . \quad . \quad (52)$$

We then have:

$$s_\alpha = -K' e a^2 c \left[\frac{\partial \phi}{\partial x_\alpha} + \frac{e}{c} P_\alpha \right]. \quad . \quad . \quad . \quad (53)$$

This expression for the world-current coincides with that suggested by Gordon * and Schrödinger.† These writers, in fact, start from a function which in our notation is

$$L \quad \sum_{\alpha=1}^{4} \left[\left(\frac{\partial \Psi}{\partial x_\alpha} + \frac{2\pi e}{hc} i P_\alpha \Psi \right) \left(\frac{\partial \bar{\Psi}}{\partial x_\alpha} - \frac{2\pi e}{hc} i P_\alpha \bar{\Psi} \right) + \frac{4\pi^2 m_0^2 c^2}{h^2} \Psi \bar{\Psi} \right], \quad (54)$$

wherein Ψ denotes the continuous wave in its complex form and $\bar{\Psi}$ the conjugate function. They then define the current four-vector by the formula

$$s_\alpha = \lambda \frac{\partial L}{\partial P_\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where λ is a constant of homogeneity: and it is easy to verify that the expressions (53) and (55) agree.

III.—The Passage from the Old Mechanics to the New

8. Lagrange's Equations in Wave Mechanics.—A glance at the general equations (37_1) and (46_1), bearing in mind the principle of the double solution and equation (47) which follows therefrom, shows that Jacobi's equation can be written in its usual form *without loss of rigour*, provided we attribute the variable proper mass

$$M_0(x, y, z, t) = \sqrt{m_0^2 - \frac{h^2}{4\pi^2 c^2} \frac{a}{a}} \quad . \quad . \quad . \quad (56)$$

to the moving body.

* *Zts. f. Phys.*, Vol. **40** (1926), p. 117.

† *Ann. der Phys.*, Vol. **82** (1927), p. 265. See also O. Klein, *Zts. f. Phys.*, Vol. **41** (1927), p. 407.

In the old dynamics the second term under the radical is neglected, which comes to the same as assuming h infinitely small.

This being so, the new mechanics can make use of Hamilton's principle and Lagrange's equations, provided the variable mass M_0 is introduced. We shall verify this, neglecting the vector potential for simplicity. Hamilton's principle will take the form

$$\delta \int_{t_0}^{t_1} L\,dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

with

$$L = -M_0 c^2 \sqrt{1-\beta^2} - F. \quad . \quad . \quad . \quad . \quad (58)$$

We are led as usual to Lagrange's equations,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v_x}\right] = \frac{\partial L}{\partial x}, \text{ &c.,} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and the components of momentum are defined by the formulæ

$$g_x = \frac{\partial L}{\partial v_x} = \frac{M_0 v_x}{\sqrt{1-\beta^2}}, \text{ &c.} \quad . \quad . \quad . \quad . \quad (60)$$

We shall still apply the name of energy to the expression (which is constant in a constant field)

$$W = \Sigma g_x v_x - L = \frac{M_0 c^2}{\sqrt{1-\beta^2}} + F. \quad . \quad . \quad . \quad (61)$$

Putting

$$\frac{\partial \phi}{\partial t} = W, \ \frac{\partial \phi}{\partial x} = -g_x, \ \frac{\partial \phi}{\partial y} = -g_y, \ \frac{\partial \phi}{\partial z} = -g_z, \quad . \quad (62)$$

and using formulæ (60) and (61), we verify the fact that we are led to Jacobi's equation,

$$\frac{1}{c^2}\left(\frac{\partial \phi}{\partial t} - F\right)^2 - \Sigma\left(\frac{\partial \phi}{\partial x}\right)^2 = M_0{}^2 c^2, \quad . \quad . \quad . \quad (63)$$

and this relation, when M_0 is defined as above, is in fact the form taken by (37_1) and (46_1) in the present case. Note also that, by combining (60) and (61), we again obtain the fundamental formula (40) for the velocity.

In short, *if we suppose that the function* $a(x, y, z, t)$ *is known*, the Lagrange-Hamilton theory enables us to calculate the form of the trajectories and the law of motion of the particles of the swarm.

Let us now make a definite statement as to the meaning of the Newtonian approximation. In this approximation, β will be so small that its square is negligible compared to unity: further, the second

term under the radical in (56) will be so small compared to the first that it is permissible to write

$$M_0(x, y, z, t) = m_0 + \epsilon(x, y, z, t), \quad . \quad . \quad (64)$$

where $\frac{\epsilon}{m_0}$ *is of the order of* β^2. Thence we obtain the approximate formulæ for the momentum and the energy:

$$g_x = m_0 v_x,\ g_y = m_0 v_y,\ g_z = m_0 v_z,\ W = m_0 c^2 + \tfrac{1}{2} m_0 v^2 + \epsilon c^2 + F, \quad (65)$$

and, whenever the absolute value of W and not variations in W are involved, we can take W as being equal to $m_0 c^2$: in particular, this justifies the passage from (40) to (40′). Finally, Lagrange's function (58) takes the approximate form

$$L = -m_0 c^2 + \tfrac{1}{2} m_0 v^2 - \epsilon c^2 - F. \quad . \quad . \quad . \quad (66)$$

Thus everything takes place as if there existed, besides F, a potential energy term ϵc^2.

IV.—The Case of the Motion of a System of Material Particles

9. **Schrödinger's Point of View.**—In his papers Schrödinger systematically considers the continuous solutions of the equations of propagation. We have in part seen how the correctness of the results thus obtained may be reconciled with the discontinuous structure of matter in the case of a single material particle.

Let us now pass to the case of an isolated system of N material particles, the proper masses of which I shall denote by $m_1, m_2, \ldots, m_N$. Confining himself to the Newtonian approximation, Schrödinger considers the configuration-space which may be constructed from the $3N$ co-ordinates $x_1, y_1, \ldots, z_N$ of the N particles, and investigates the propagation of a wave in this hyperspace. According to Schrödinger, the propagation would take place in accordance with the equation:

$$\sum_1^N \frac{1}{m_i}\left[\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} + \frac{\partial^2 u}{\partial z_i^2}\right] + \frac{8\pi^2}{h^2}(E - F)u = 0, \quad (67)$$

where E is the total energy in the Newtonian sense and $F(x_1, \ldots, z_N)$ the potential energy function.

This appears to be a natural assumption, for equation (19), which is valid for one particle in a constant field, may to the Newtonian approximation be written, bearing in mind the form of u and dropping the suffix of the proper mass,

$$\frac{1}{m}\nabla^2 u + \frac{8\pi^2}{h^2}[E - F]u = 0, \quad . \quad . \quad . \quad (67')$$

a form of which (67) is in fact the generalization.

But equation (67) raises two difficulties. To begin with, according to Schrödinger's ideas the material particle formed by a group of waves would not have the character of a point-singularity, and in micromechanics we could no longer speak of its position or of its trajectory. But, in that case, what would be the meaning of the co-ordinates $x_1, \ldots, z_N$, out of which the abstract configuration-space is built up? This difficulty disappears if it is assumed, as we have done, that the material particle is always quite definite.

There is another difficulty, however. Propagation in a configuration-space of purely abstract existence is, in fact, out of the question from the physical point of view. The wave representation of our system ought to involve N waves propagated in real space instead of a single wave propagated in the configuration-space. What, then, does Schrödinger's equation really mean? That is what we must try to find out.

10. **The Meaning of Equation (67).**—For simplicity we shall consider an isolated system consisting of two material particles, as the extension of the reasoning to the case of N particles presents no theoretical difficulty. For us, each particle constitutes a singularity in a wave phenomenon in space. Neglecting magnetic actions, the propagation of the two waves takes place in accordance with the equations:

$$\left.\begin{aligned} \Box u_1 + \frac{4\pi i}{h}\frac{F_1}{c^2}\frac{\partial u_1}{\partial t} - \frac{4\pi^2}{h^2}\left[m_1^2 c^2 - \frac{F_1^2}{c^2}\right]u_1 = 0 \\ \Box u_2 + \frac{4\pi i}{h}\frac{F_2}{c^2}\frac{\partial u_2}{\partial t} - \frac{4\pi^2}{h^2}\left[m_2^2 c^2 - \frac{F_2^2}{c^2}\right]u_2 = 0 \end{aligned}\right\}. \qquad (68)$$

The variables (x, y, z), which indicate any point in space, must be carefully distinguished from (x_1, y_1, z_1) and (x_2, y_2, z_2), the co-ordinates of the two particles. In accordance with the principle of action and reaction, we shall assign the following forms to the potential functions $F_1(x, y, z, x_2, y_2, z_2)$ and $F_2(x, y, z, x_1, y_1, z_1)$:

$$\left.\begin{aligned} F_1 = F(\sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2}) \\ F_2 = F(\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}) \end{aligned}\right\}, \qquad (69)$$

so that the value of F_1 at the point occupied by the first moving particle is equal to the value of F_2 at the point occupied by the second: if the distance between the two particles is r, this value is $F(r)$. The propagation in space of either wave thus depends at every point on the value of the potential which corresponds to the position at that time of the singularity of the other wave.

For each of the equations (68) we must find a solution involving a singularity such that both relations are satisfied. Let us consider,

as Schrödinger did, the Newtonian approximation. In the old mechanics there exists a Jacobi function of the system $\phi(x_1, \ldots, z_2)$ such that the momenta are:

$$m_1 v_{1x} = -\frac{\partial \phi}{\partial x_1}, \text{ &c.: } m_2 v_{2x} = -\frac{\partial \phi}{\partial x_2}, \text{ &c.} \quad . \quad (70)$$

Can a similar function ϕ be defined in the Newtonian approximation of the new mechanics? For the moment we shall consider the motion of the second material particle to be known: the motion of the first then takes place in a field which is a known function of (x, y, z, t), a case which we have examined. If the initial velocity of the first particle is given we know that the series of its possible motions is represented by a wave with continuous amplitude

$$a_1(x, y, z, t)\, e^{\frac{2\pi i}{h}\phi_1(x, y, z, t)}.$$

By the last section but one, the equations of motion can then be written in the Lagrangian form

$$\frac{d}{dt}\left[\frac{\partial L_1}{\partial v_{1x}}\right] = \frac{\partial L_1}{\partial x_1}, \text{ &c.,} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where
$$L_1 = \tfrac{1}{2} m_1 v_1^2 - \epsilon_1(x_1, y_1, z_1, t)\, c^2 - F(r). \quad . \quad (72)$$

Similarly, if we consider the motion of the first particle to be known, the motion of the second will be determined by the equations

$$\frac{d}{dt}\left[\frac{\partial L_2}{\partial v_{2x}}\right] = \frac{\partial L_2}{\partial x_2}, \text{ &c.,} \quad . \quad . \quad . \quad . \quad . \quad (73)$$

with
$$L_2 = \tfrac{1}{2} m_2 v_2^2 - \epsilon_2(x_2, y_2, z_2, t)\, c^2 - F(r). \quad . \quad (74)$$

Equations (71) and (73) have to be solved simultaneously. In classical mechanics it is possible to find a Lagrangian function L for the whole system such that equations (71) and (73) may be written:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial q'}\right] = \frac{\partial L}{\partial q} \qquad \left(q' = \frac{dq}{dt}\right), \quad . \quad . \quad . \quad . \quad (75)$$

where q is any one of the six variables $x_1, \ldots, z_2$: and we know that it is then possible to define a Jacobi function $\phi(x_1, \ldots, z_2)$ which satisfies the relations (70). I have shown elsewhere * that, in order that it may be possible to obtain this function L, it must be possible to separate the terms in L_1 and L_2 which depend on mutual actions from those which do not. This separation being supposed to have been carried out, we take for our function L the sum of the terms of the second kind added to half the sum of the terms of the first

* *Ondes et Mouvements*, p. 43 and following pages.

kind. We can proceed in this way in classical mechanics because in L_1 and L_2 the terms in ϵ_1 and ϵ_2 are neglected and we are thus enabled to write

$$L = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 - F(r). \quad . \quad . \quad . \quad (76)$$

For the same to be true in the new mechanics, ϵ_1 and ϵ_2 must reduce to the *same* function of the distance r: in other words, the extra potential energy terms introduced by the new conceptions must have the same mutual character as those defined by the functions F_1 and F_2. This is the natural extension of the principle of action and reaction. If we assume this principle, we can form the Lagrangian function of the system by adding to the second member of (76) the new mutual term $-\epsilon(r)c^2$ and we shall thence deduce, as usual, the existence of a function $\phi(x_1, \ldots, z_2)$ satisfying equations (70).

As for us the material particles have quite definite co-ordinates, we shall be able to build up a configuration-space without ambiguity. The system of the two moving particles will be represented therein by a point whose six components of velocity will be given by the equations (70). Let us still suppose that the initial velocities are given but not the initial positions; to the different assumptions that we can make as to these initial positions there will correspond different trajectories of the representative point, and to the series of all the possibilities considered as occurring simultaneously there will correspond a swarm of representative points. The motion of the swarm is steady and satisfies the equation of continuity,

$$\operatorname{div}(\rho \mathbf{v}) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

where $\rho(x_1, \ldots, z_2)$ is the density of the swarm and $\mathbf{v}$ its velocity. Taking account of (70), this equation becomes, in notation whose meaning is obvious,

$$\sum_{x,y,z}\left[\frac{1}{m_1}\frac{\partial\phi}{\partial x_1}\frac{\partial(\log\rho)}{\partial x_1} + \frac{1}{m_2}\frac{\partial\phi}{\partial x_2}\frac{\partial(\log\rho)}{\partial x_2}\right] + \frac{1}{m_1}\nabla_1^2\phi + \frac{1}{m_2}\nabla_2^2\phi = 0. \quad (78)$$

Now, if we consider Schrödinger's equation (67) and look for a continuous solution of the form:

$$\psi(x_1, y_1, z_1, x_2, y_2, z_2, t) = A(x_1, \ldots, z_2)e^{\frac{2\pi i}{h}\phi}, \quad . \quad (79)$$

we find by substitution that A must satisfy the relation

$$\Sigma\left[\frac{1}{m_1}\frac{\partial\phi}{\partial x_1}\frac{\partial(\log A^2)}{\partial x_1} + \frac{1}{m_2}\frac{\partial\phi}{\partial x_2}\frac{\partial(\log A^2)}{\partial x_2}\right] + \frac{1}{m_1}\nabla_1^2\phi + \frac{1}{m_2}\nabla_2^2\phi = 0. \quad . \quad . \quad . \quad (80)$$

According to (78) and (80), A, the amplitude of the fictitious wave (79), will thus play the same part here as the amplitude of the continuous wave plays in the case of a single particle: in other words, the product $A^2 d\tau$ will at each point of the configuration-space be a measure of the probability of the presence of the representative point within the element of volume $d\tau$.

This conclusion is confirmed by the following observation: if the two particles have no mutual action, Schrödinger's equation admits as a solution the *product* of the continuous functions Ψ corresponding to the two particles, and since the probabilities of presence of the two points are in this case quite independent, this agrees as it ought with the theorem of composite probabilities.

To sum up: (1) Schrödinger's equation has a meaning only if it is possible to construct a configuration-space, that is, if the particles have a well-defined position in space. (2) This equation is not really a physical equation of propagation, but in the square of the amplitude of the appropriate solution it furnishes the probability that the system should be in a given state, when the initial positions of its constituents are unknown.

I must add that it seems difficult to find an equation analogous to (67) if one does not wish to limit oneself to the Newtonian approximation.

Fermi's elegant calculation with reference to the scattering of electrons by a rotator may be regarded as an illustration of the foregoing.*

V.—Summary and Remarks

11. The Guiding Wave.—If the series of results obtained in Section II is examined, we see that the results are summarized by the two fundamental formulæ (40) and (49):

$$\begin{cases} \mathbf{v} = -c^2 \dfrac{\operatorname{grad}\phi + \dfrac{e}{c}\mathbf{A}}{\dfrac{\partial \phi}{\partial t} - e\mathcal{V}} \quad \ldots \text{(I)} \\ \rho(x, y, z, t) = C \times a^2 \left(\dfrac{\partial \phi}{\partial t} - e\mathcal{V} \right), \quad \ldots \text{(II)} \end{cases}$$

where C is a constant.

I reached the first of these relations (and the second is a consequence thereof) by way of the principle of the double solution. This principle is verifiable in the case of no field, but remains hypothetical in the general case. In my opinion, it is necessary even in micro-

* *Zts. f. Phys.*, Vol. **40** (1926), p. 399.

mechanics to keep the idea of the atomicity of matter, if it were only to give a meaning to Schrödinger's equation (67). But if we do not wish to appeal to the principle of the double solution, it is permissible to adopt the following point of view: assume the existence of the material particle and of the continuous wave represented by the function Ψ as distinct realities, and postulate that the motion of the particle is determined as a function of the phase of the wave by equation (I). The continuous wave is then thought of as directing the motion of the particle: it is a guiding wave.

By thus postulating equation (I) we avoid having to justify it by the principle of the double solution; but this, I believe, can be only a provisional attitude. The corpuscle will doubtless have to be *reincorporated* into the wave phenomenon, and we shall probably be led back to ideas analogous to those developed above.

I shall indicate two of the most important applications of the formulæ (I) and (II).

In the case of light, the continuous wave Ψ is that considered in classical optics, and since, from (II), the density of the photons is proportional to the square of the amplitude, the phenomena of undulatory optics will be predicted equally well by the old and by the new theory of light.

Our fundamental formulæ also seem to lead to the justification of one of Schrödinger's hypotheses. Consider a collection of atoms of hydrogen, the state of which is defined, from Schrödinger's point of view, by the same function Ψ, a sum of fundamental functions. For us, the electron has a quite definite position and velocity in each atom, but if we superpose all the atoms in imagination, we shall obtain a sort of average atom where the electrical density is, from (II), clearly given, to the Newtonian approximation, by

$$\delta = e\rho = Ca^2 = C\Psi\overline{\Psi},$$

where C is a constant.

This is, in fact, the expression proposed by Schrödinger: here it appears as the definition of a sort of average density.

Finally, it is easy to obtain from this point of view all the formulæ of Heisenberg's theory of matrices in the form given them by Schrödinger.

12. **The Constrained States of the Material Particle.**— In the paper already quoted, Schrödinger gave the expression for the energy-momentum tensor corresponding to the continuous waves Ψ. If the precise meaning stated above is attributed to the continuous waves, this tensor can be decomposed into a tensor giving the energy and momentum of the particles and a tensor corresponding to stresses which exist in the wave phenomenon surrounding the particles. These stresses vanish in the mechanical states which are

in accordance with the old dynamics: they are characteristic of the new states predicted by the wave mechanics [those of formulæ (13), for example], which now appear as constrained states of the particle.

This observation enables us to remove a difficulty in connexion with the pressure exerted on a boundary surface by a flow of corpuscles. As a rule this pressure is calculated on the supposition that the corpuscles rebound from the surface and communicate a certain impulse to it by the blow: this is the way in which the pressure of a gas is calculated in the kinetic theory or that of black radiation in the corpuscular theory of light.

According to the wave mechanics, however, there exists in the neighbourhood of the surface a state of interference due to the superposition of the incident and reflected waves, and the application of formula (1) shows that the particles do not actually strike the boundary. How is it, then, that there is a pressure on the surface? This can only come from the stresses which exist in the region of interference. On account of these stresses, the same pressure should be exerted on the boundary surface as if the particles communicated an impulse to it by rebounding off it; and this can in fact be demonstrated by calculation from Schrödinger's formulæ.

Appendix Added by the Author for this Edition

The results of the above paper can easily be put in a more general form which includes the case of gravitational fields.

We shall start from the general equation given by de Donder for a particle with proper mass m_0 and electric charge e:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left[\sqrt{g}\,g^{ik}\frac{\partial\psi}{\partial x^i}\right] - \frac{4\pi ie}{h}A^k\frac{\partial\psi}{\partial x^k} + \frac{4\pi^2}{h^2}[m_0^2c^2 - eA^2]\psi = 0. \quad (1)$$

The quantities A^k are the components of the four-dimensional vector electromagnetic potential, and $A^2 = g_{ik}A^iA^k$ is the square of its length. The operator

$$-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}\left[\sqrt{g}\,g^{kl}\frac{\partial}{\partial x^k}\right] \quad (11)$$

is the generalization for non-Euclidean space-time of the usual Dalembertian.*

* [T. Levi Civita, *Absolute Differential Calculus*, p. 154.]

Let us put $$M_0 = \sqrt{m_0 - \frac{h^2}{4\pi^2 c^2} \cdot \frac{\Box a}{a}}, \quad . \quad . \quad . \quad . \quad (2)$$

and suppose that ψ is of the form

$$\psi = a(x_1, x_2, x_3, x_4) \cos \frac{2\pi}{h} \phi(x_1, x_2, x_3, x_4),$$

a and ϕ being real continuous functions. We find, by substitution in (1):

$$g^{kl}\left(\frac{\partial \phi}{\partial x^k} - eP_k\right)\left(\frac{\partial \phi}{\partial x^l} - eP_l\right) = M_0^2 c^2, \quad . \quad . \quad (3)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^l}\left[\sqrt{-g}\, g^{kl} a^2 \left(\frac{\partial \phi}{\partial x^k} - eP_k\right)\right] = 0. \quad . \quad (4)$$

Equation (3) is the classical Jacobi equation *in which the constant proper mass m_0 is replaced by the corrected variable proper mass M_0.* We are thus naturally led to define the four-dimensional velocity of the particles in a swarm linked with the same ψ-wave by the relation:

$$M_0 c\, u^l = g^{kl}\left(\frac{\partial \phi}{\partial x^k} - eP_k\right) \qquad u^l = \frac{dx^l}{ds}. \quad . \quad . \quad (5)$$

According to (3), this definition is in agreement with the equation:

$$u_l u^l = 1,$$

which follows from the definition of the u^l's.

Equation (4) shows us that the vector, the covariant components of which are

$$a^2\left(\frac{\partial \phi}{\partial x^k} - eP_k\right),$$

has zero divergence. We can therefore assume that this vector is proportional to the vector

$$C^l = \rho_0 u^l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which represents the " stream of particles " in the swarm. Consequently, we shall write:

$$\rho_0 = K M_0 c a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The number of particles per unit volume is given by the fourth component of C^l:

$$\rho = C^4 = K M_0 c\, a^2 u^4 = K a^2 g^{4k}\left(\frac{\partial \phi}{\partial x^k} - eP_k\right). \quad . \quad (8)$$

If there is no gravitational field, the formulæ (5) and (8) are merely the fundamental formulæ (I) and (II) of the above paper.

Summing up, we can say: equation (3) shows that the ψ-wave is a *guiding wave* by which the motion of the particles is controlled, whereas equation (4) shows that the ψ-wave is also a *probability wave* giving the probability of presence of a particle in an element of volume when the initial position is unknown.

It is easy to show that the equations of motion for a particle can be written in the form:

$$\frac{d}{ds}(M_0 cu_l) = \tfrac{1}{2} M_0 cu^i u^k \frac{\partial g_{ik}}{\partial x^l} + eu^i \left(\frac{\partial P_i}{\partial x^l} - \frac{\partial P_l}{\partial x^i}\right) + c\frac{\partial M_0}{\partial x^l}. \quad (9)$$

The last term of the second member is characteristic of the new mechanics, and may be compared to a kind of fictitious force.

Let us now suppose that there is no gravitational field and again consider a swarm of particles linked with the same ψ-wave. Multiplying (9) by $M_0 a^2$, we get, after some transformations,

$$\frac{\partial}{\partial x^k}[T_i^k + \Pi_i^k + S_i^k] = 0, \quad . \quad . \quad . \quad . \quad (10)$$

where the quantities S_i^k are the mixed components of the electromagnetic energy-stress tensor. T_i^k and Π_i^k are defined by the relations

$$\left.\begin{aligned} T_i^k &= \rho_0 M_0 u^k u_i \\ \Pi_i^k &= K\frac{h^2}{8\pi^2 c} g^{lk}\left[2\frac{\partial a}{\partial x^i}\frac{\partial a}{\partial x^l} - g_{il}\left(g^{mn}\frac{\partial a}{\partial x^m}\frac{\partial a}{\partial x^n} - a \,\square\, a\right)\right] \end{aligned}\right\}. \quad (11)$$

The quantities T_i^k are then the mixed components of the usual energy-momentum tensor of the swarm of particles, but the tensor Π corresponds to an " internal pressure " which was unknown to the old mechanics. Here we do not have conservation of energy and momentum for the particles and the electromagnetic field in general. The occurrence of the tensor Π is intimately connected with the variability of the corrected proper mass M_0 and is of great importance in the new dynamics.

A Comparison of the Different Statistical Methods Applied to Quantum Problems

BY

LÉON BRILLOUIN

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Summary.—The writer recalls the essential points of three statistical theories which have been developed in connexion with quantized phenomena. In *classical statistics*, it is assumed that the objects to be distributed (atoms, electrons, or photons) can be distinguished among themselves *a priori*, although their individual properties are similar; the projectiles are also assumed to be independent, so that they do not exert an influence on each other. The *new statistics* is based, on the contrary, on the assumption that the projectiles are completely identical and cannot be distinguished among themselves. Bose and Einstein suppose that the projectiles are independent, but Fermi and Dirac assume the existence of a mutual influence characterized by the fact that more than one projectile cannot occupy a single cell of phase extension.

This identity between the objects is certainly logical in principle; but in practice it corresponds to a process which cannot be physically realized, for all the projectiles are supposed to be thrown at random into the different cells at a given instant. In reality things happen in quite a different way; the atoms of a gas move from one compartment to another as a result of collisions. If they are identical *a priori*, it is easy, nevertheless, to distinguish them by their history.

The writer therefore assumes that different projectiles can be distinguished, and tries to find what auxiliary assumption will enable the results of any of the three statistical theories mentioned above to be obtained at will. This assumption will involve the laws of accumulation of the projectiles in a single compartment. If each object completely fills a cell, the formulæ of Fermi and Dirac are obtained. When the projectiles are assumed to be independent, the classical expressions naturally result; to obtain formulæ of the Bose-Einstein type, it must be assumed that the probability that a projectile lands in a compartment which already contains p projectiles is proportional to $1 + p$.

This curious assumption leads rigorously to the laws of emission and absorption of light stated by Einstein, as well as to all the generalizations of them; the fluctuations of black radiation are also correctly obtained.

1. **Introduction.**—In the last few years, various statistical methods have been developed in connexion with quantum problems; these methods are equally applicable to the study of material systems and to the investigation of the distribution of radiation. I shall first briefly summarize the different points of view, on the lines of a recent

account by M. Langevin; I shall then show that the same ultimate formulæ can be obtained by starting from widely differing initial assumptions. Thus the agreement of a given formula with facts is not sufficient as a justification of the postulates.

The problem always reduces to that of the distribution of counters among a certain number of compartments. These counters are the molecules if we are studying a gas, or light quanta $h\nu$ in the case of radiation. We define the actual state of each of these projectiles by three co-ordinates of position (x_1, x_2, x_3) and the three momenta p_1, p_2, p_3; p, the total momentum of the projectile, is given by

$$p^2 = p_1^2 + p_2^2 + p_3^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The projectile is compelled to remain in an enclosure of volume V; its energy, which is purely kinetic, is given by the formulæ

$$E = \frac{p^2}{2m} \text{ (for the molecule } m\text{) or } E = h\nu = pc \text{ (for the photon } h\nu\text{).} \quad (2)$$

In the phase extension (a space of six dimensions $x_1, x_2, x_3, p_1, p_2, p_3$) the instantaneous state of a projectile is represented by a point. The quantum conditions oblige us to assume that this extension is composed of cells of finite volume h^3. Two representative points situated in the same cell are supposed to be indistinguishable; they correspond to two projectiles with the same co-ordinates and the same momenta, whose motions will proceed in the same way.

The shape of the cells is unknown to us, but this will not inconvenience us in practice. We shall have to evaluate g, the number of cells included in a volume Φ of phase extension; if Φ is very large compared with h^3, we shall be content with the value

$$g = \frac{\Phi}{h^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For example, let us consider the projectiles for which the positions are arbitrary (within V) and for which the momentum lies between p and $p + \Delta p$; Δp will be very small compared to p, but large compared to the dimensions of a cell.

Φ, the volume of phase extension, is obtained by multiplying V, the volume of the configuration extension, by the volume of the momentum extension; in this latter extension the above condition defines a spherical layer of radius p and thickness Δp. Since this layer has volume $4\pi p^2 \Delta p$, it follows that

$$\Phi = V \cdot 4\pi p^2 \Delta p, \quad g = V \frac{4\pi p^2}{h^3} \Delta p. \quad . \quad . \quad . \quad (4)$$

To say that the momentum lies between p and $p + \Delta p$ means

that the energy of the particle lies between E and $E + \Delta E$; according as we are dealing with material particles or photons, we use one or other of the relations (2), and we find

$$g = \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} E^{\frac{1}{2}} \Delta E \text{ for the gas,} \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

$$g = V\frac{4\pi E^2}{h^3 c^3}\Delta E = V\frac{4\pi\nu^2}{c^3}\Delta\nu \text{ for the photons.} \quad . \quad (5b)$$

In the latter case we assume that the result must be doubled in order to take account of the two ways in which light may be polarized.

2. **The Various Statistical Theories.**—The statistical problem is then as follows: in how many different ways can a certain number of projectiles be arranged in compartments, at the rate of n_i per layer of energy E_i, where g_i, the number of compartments in each layer, is given by the formulæ (5)?

The answer will depend essentially on the method employed for the counting; two distinct definitions are involved: (1) that of the different cases considered possible; (2) the criterion used to distinguish different arrangements.

If we follow the methods of *classical statistics* we assume that one projectile can be distinguished from another *a priori*. Let there be N projectiles $a_1, a_2, \ldots, a_N$, and a certain number of compartments $b_1, b_2, \ldots, b_g$; if we put the object a_1 into the compartment b_1, and a_2 into b_2, we have an arrangement distinct from that obtained by putting a_1 into b_2 and a_2 into b_1. From the point of view of the different cases considered possible, the objects are usually assumed independent of each other; i.e. the chances are equal that we add an object to a compartment which is already occupied, or that we place it in an empty compartment.

In the *new statistics*,* all the projectiles are considered to be identical *a priori*; the two arrangements distinguished above will only count as one: an object a in b_1 and another in b_2. The arrangements are distinguished only by the total number of objects placed in each compartment. To this convention must be added that of the possible cases: according to Bose and Einstein the objects are independent, and an object may equally well be added to an occupied compartment or placed in an empty compartment. According to Pauli, Dirac, and Fermi, each object fills a compartment, and more than one object cannot be placed in a compartment.

* S. N. Bose, *Zts. f. Phys*, Vol. **26** (1924), p. 178; Vol. **27** (1924), p. 384; A. Einstein, *Berl. Acad.* (1924), p. 261; (1925), p. 3 and p. 18; E. Schrödinger, *Phys. Zts.*, Vol. **27** (1926), p. 95; E. Fermi, *Nuovo Cim.*, Vol. **1** (1924), p. 145; *Atti Lincei* (6), Vol. **3** (1926), p. 145; *Zts. f. Phys.*, Vol. **36** (1926), p. 902; W. Heisenberg, *Zts. f. Phys.*, Vol. **38** (1926), p. 411; P. A. M. Dirac *Proc. Roy. Soc.*, Vol. **112** (1926), p. 661.

It is extremely easy to apply these definitions. To begin with, let us consider the *classical definitions*; we have to arrange N distinct objects at the rate of n_i per layer E_i, where the number of compartments in the layer E_i is g_i. We first find in how many different ways we can form the groups $n_1, n_2, \ldots, n_i, \ldots$ which will subsequently be found in each layer. The number of ways is

$$\frac{N!}{n_1!n_2!\ldots n_i!\ldots} = \frac{N!}{\prod_i (n_i!)}. \quad \ldots \quad (6)$$

Inside the layer E_i we now have to arrange n_i distinct objects in g_i compartments, and each object may be placed in any of the compartments; this gives $g_i^{n_i}$ arrangements. Combining these two results, we obtain:

$$W = N! \frac{\prod_i g_i^{n_i}}{\prod_i (n_i!)}, \quad \ldots \quad (7)$$

or, by Stirling's formula:

$$\log W = N \log N + \sum_i [-n_i \log n_i + n_i \log g_i]. \quad (8)$$

In order to find the most probable arrangement, the variation of $\log W$ for a series of variations δn_i must be made to vanish, while N, the total number of objects, and U, the total energy, given by

$$N = \sum_i n_i, \quad U = \sum_i n_i E_i, \quad \ldots \quad (9)$$

remain constant. The term in $N \log N$ in formula (8) is therefore constant and of no importance.

Now let us assume *identity of the projectiles*; the division of the objects into groups $n_1, n_2, \ldots, n_i, \ldots$ assigned to the different layers can be made in only one way. How is the distribution within a single layer subsequently carried out?

According to Pauli and Dirac, we can place only one object in a compartment. Thus we have to find the number of combinations of g_i things n_i at a time, without repetitions. This number is

$$w_i = \frac{g_i!}{n_i!\,(g_i - n_i)!}. \quad \ldots \quad (10)$$

For all the layers, we obtain

$$W = \prod_i w_i = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!}, \quad \ldots \quad (11)$$

and by Stirling's formula

$$\log W = \sum_i g_i \log g_i + \sum_i [-n_i \log n_i + (n_i - g_i) \log (g_i - n_i)]. \quad (12)$$

In the process of finding the maximum of $\log W$, the first term is constant, since g_i is given; and N, the total number of objects, does not appear in the formula.

If we assume the definitions of Bose and Einstein, we can place any number of objects in each compartment. Thus we reckon up the number of combinations of g_i things n_i at a time, *repetitions being allowed*; we deduce the formulæ:

$$w_i = \frac{(g_i + n_i - 1)!}{n_i!\,(g_i - 1)!}, \quad \ldots\ldots \quad (13)$$

$$W = \prod_i w_i = \prod_i \frac{(g_i + n_i - 1)!}{n_i!\,(g_i - 1)!}. \quad \ldots \quad (14)$$

$$\log W = -\sum_i g_i \log g_i + \sum_i [-n_i \log n_i + (n_i + g_i)\log(g_i + n_i)]; \quad (15)$$

since n_i and g_i are very large numbers, $g_i - 1$ and g_i may be taken as equal.

This last statistical theory is the only one of the three described here that leads, in the case of radiation, to Planck's well-known formula.

The three formulæ above may be summarized in a single form, if constant terms of no importance are disregarded:

$$\log W = A - \sum_i n_i \log n_i + \begin{cases} \sum_i (n_i - g_i)\log(g_i - n_i) \ldots \text{Pauli-Dirac} & (a) \\ \sum_i n_i \log g_i \qquad\qquad \ldots \text{classical} & (b) \\ \sum_i (n_i + g_i)\log(g_i + n_i) \ldots \text{Bose-Einstein} & (c) \end{cases} \quad (16)$$

3. **Discussion; the Investigation of a Single Theory.***—It is curious to find that such different definitions lead to such nearly identical formulæ; if the different statistical theories are applied to gases, the differences are minute, and they could only appear in the conditions known as those of *degenerate gases*, i.e. gases at very low temperature and very high density.

I wished to investigate whether formulæ of type (16) could be obtained from a single theory, and to begin with I assumed that the projectiles were identical. Since the classical formula is intermediate between that of Bose (with independent projectiles) and that of Pauli (with exclusion), it would therefore seem that the classical formula can be found by assuming an intermediate law. The probability that a pth projectile should be thrown into a compartment which already

* L. Brillouin, *C. R.*, Vol. **184** (1927), p. 589.

contains $(p-1)$ would be a function which decreases rapidly as p increases, but the results obtained in this way are not simple.*

For reasons given later in § 6, it also seemed to me that the assumption of the identity of projectiles was inconvenient and represented a development which was unrealizable in practice. When we wish to analyse an atomic mechanism, we shall consider the conditions satisfied by a given atom or quantum in order to forecast their subsequent motion.

I shall therefore retain the assumption of the classical theory, that the projectiles are quite distinct from each other, although they have similar properties. I shall assume, however, that the projectiles are not independent of each other, so that the probability that an object will land in a compartment already occupied will depend on the number of previous occupants. With regard to this, I shall make the following very simple assumption: each compartment has capacity 1 when empty, and each projectile has volume a; a compartment which contains one projectile has still space $1-a$ at its disposal, while one which contains p projectiles has free space $1-pa$.

I shall follow the line of argument indicated above in connexion with the classical theory, and I shall begin by dividing my N projectiles into parcels $n_1, n_2, \ldots, n_i, \ldots$ which will be assigned to the layers of energy $E_1, E_2, \ldots, E_i, \ldots$. The number of these arrangements will still be given by formula (6).

Now I have to distribute the n_i objects among the g_i compartments of the level E_i. The number of ways of placing the first object is g_i; for the second, $g_i - a$ places are available; and for the pth, $g_i-(p-1)a$ places. The number of possible arrangements is therefore:

$$\left.\begin{aligned}\omega_i &= g_i(g_i-a)(g_i-2a)\ldots(g_i-(n_i-1)a)\\ &= a^{n_i}\frac{g_i}{a}\left(\frac{g_i}{a}-1\right)\ldots\left(\frac{g_i}{a}-n_i+1\right)\end{aligned}\right\}, \quad . \quad (17)$$

or, using the properties of Γ functions:

$$\omega_i = a^{n_i}\frac{\Gamma\left(\frac{g_i}{a}+1\right)}{\Gamma\left(\frac{g_i}{a}-n_i+1\right)}. \quad . \quad . \quad . \quad . \quad (18)$$

The total number of arrangements with n_i projectiles in the layer

* We are led to assume that a compartment containing p projectiles has a probability equal *a priori* to $\frac{1}{p!}$.

E_i is therefore obtained by multiplying (6) by (18), which gives

$$N!\prod_i \frac{a^{n_i}}{n_i!}\,\frac{\Gamma\left(\frac{g_i}{a}+1\right)}{\Gamma\left(\frac{g_i}{a}-n_i+1\right)}, \text{ where } N=\sum_i n_i.$$

To find the chance that such an arrangement should occur, we must divide this result by the total number of possible events; the latter number is obviously

$$G(G-a)(G-2a)\ldots(G-(N-1)a)=a^N\,\frac{\Gamma\left(\frac{G}{a}+1\right)}{\Gamma\left(\frac{G}{a}-N+1\right)},$$

where $G(=\sum_i g_i)$ is the total number of compartments in all the levels. Thus I finally obtain the probability:

$$W=\frac{N!\,\Gamma\left(\frac{G}{a}-N+1\right)}{a^N\,\Gamma\left(\frac{G}{a}+1\right)}\prod_i \frac{a^{n_i}}{n_i!}\,\frac{\Gamma\left(\frac{g_i}{a}+1\right)}{\Gamma\left(\frac{g_i}{a}-n_i+1\right)}. \qquad (19)$$

The numbers n and g are large in practice, so that I can apply Stirling's formula and replace $\Gamma(x)$ by x^x; this gives

$$W=\frac{N^N(G-aN)^{\frac{G}{a}-N}}{G^{\frac{G}{a}}}\prod_i \frac{g_i^{\frac{g_i}{a}}}{n_i^{n_i}(g_i-an_i)^{\frac{g_i}{a}-n_i}}. \qquad (20)$$

Hence

$$\left.\begin{aligned}\log W=&\frac{1}{a}\left[\sum_i g_i\log g_i-G\log G\right]\\&+N\log N+\left(\frac{G}{a}-N\right)\log(G-aN)\\&+\sum_i\left[-n_i\log n_i+\left(n_i-\frac{g_i}{a}\right)\log(g_i-an_i)\right]\end{aligned}\right\}. \qquad (21)$$

I shall first assume that N, the total number of projectiles, is a constant, as are also the g_i's; it is clear that my general formula is of type (16) and that it will enable me to obtain the various particular cases previously considered by giving different values to a.

For this, I should have to make the following assumptions:

1. *The Pauli-Dirac assumption*: $a=1$.

Each object completely fills a compartment and leaves no room for other objects; a formula of type (16*a*) is obtained.

2. *The classical assumption*: *the projectiles independent of each other*.

To get this case it is sufficient to put $a = 0$.

3. *The assumption of attraction between the projectiles*: $a = -1$.

With this assumption, the probability that a projectile should land in a compartment which already contains p of them is equal to $1 + p$. In this case a formula similar to the Bose-Einstein formula is obtained.

4. **Discussion; the Case of Black Radiation.**—The three formulæ (16), or their equivalent (21), give practically the same results when they are applied to the distribution of the molecules of a gas; this is the celebrated problem of degeneration of gases, which has led to wonderful results when applied to the problem of the free electrons in metals.*

In the case of radiation, as is known, Planck's law can be deduced from the Bose-Einstein formula only. My formula is not quite the same, since N, the total number of projectiles, appears in it, so that it is worth while to examine the process of deduction more closely. I accordingly suppose that the projectiles are photons $h\nu$, and I wish to find their most probable distribution among the various levels of energy. I take the total energy as given by

$$U = \sum_i n_i E_i = \sum_i n_i h\nu_i, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and I leave N, the total number of projectiles, undetermined. With a variation δn_i of the numbers n_i, I shall have a variation $\delta N = \sum_i \delta n_i$ in the total number of projectiles. The quantities g_i, which are the numbers of compartments in each layer, are constants; G, the total number of compartments, is a number *which, though extremely great, is finite*, for it is clear that levels of energy E greater than my total energy U cannot be attained.

What will the variation of $\log W$ be?

$$\delta \log W = \delta N\,[\log N - \log(G - aN)] + \sum_i \delta n_i[-\log n_i + \log(g_i - an_i)] = \sum_i \delta n_i \log \frac{N(g_i - an_i)}{n_i(G - aN)}. \quad (23)$$

In order to find the most probable distribution, I must equate $\delta \log W$ to zero, with the auxiliary condition

$$\delta U = \sum_i E_i \delta n_i = 0. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

* W. Pauli, *Zts. f. Phys.*, Vol. **41** (1927), p. 81; A. Sommerfeld, *Naturwiss.*, Vol. **41** (1927), p. 825.

Hence, if I multiply the latter equation by $-\beta$ and add it to the former, I must have

$$\delta \log W - \beta \delta U = \sum_i \delta n_i \left[\log \frac{N(g_i - an_i)}{n_i(G - aN)} - \beta E_i\right] = 0. \quad (25)$$

Equating each of the coefficients of the quantities δn_i to zero, I obtain the relations:

$$\frac{N(g_i - an_i)}{n_i(G - aN)} = e^{\beta E_i}, \text{ or } n_i = \frac{g_i}{\frac{G - aN}{N} e^{\beta E_i} + a}. \quad (26)$$

This formula shows that I must adopt the third assumption of last section ($a = -1$) in order to obtain Planck's expression. What does $\frac{G - aN}{N}$, the numerical coefficient of the exponential, then become? The number G is very great, but finite; on the contrary, N, the number of projectiles available, is infinite; indeed, the only limitation which I imposed was that of the value U of the total energy. I can therefore keep on accumulating projectiles indefinitely in the compartment of zero energy, this compartment being on my assumption capable of perpetual expansion. Thus in the limit I must take

$$\frac{G - aN}{N} = \frac{G + N}{N} = 1,$$

whence

$$n_i = \frac{g_i}{e^{\beta E_i} - 1} = V \frac{8\pi \nu_i^2 d\nu}{c^3} \frac{1}{e^{\beta h\nu_i} - 1}, \quad (27)$$

using the value of g_i given in (5*b*). The energy of radiation of frequency ν_i (to $d\nu_i$ *près*) is found by multiplying this expression by $h\nu_i$. Thus I obtain Planck's law; the coefficient β is proved equal to $\frac{1}{kT}$ in the well-known way.

5. **Fluctuations in Radiation.**—It is not without interest to see that special laws of attraction (or coupling) between the photons yield a simple explanation of fluctuations in radiation. This is a very important problem, which was raised by Einstein, and which has attracted the attention of numerous investigators.*

Let m_i be a deviation from $\bar{n}_i$, the most probable number of particles allotted to the level of energy E_i, so that

$$n_i = \bar{n}_i + m_i.$$

* A. Einstein, *Phys. Zeits.*, Vol. **10** (1909), p. 189; H. A. Lorentz, *Les théories statistiques en thermodynamique* (Leipzig, Teubner, 1916); W. Bothe, *Zts. f. Phys.*, Vol. **20** (1923), p. 145; Vol. **23** (1924), p. 214; Vol. **41** (1927), p. 345.

In this last article, Bothe finds the fluctuation-formula by means of assumptions very like mine, but his indirect proof is not very satisfactory.

What will be the probability that this deviation should occur? We shall get it by finding an expansion in Taylor's series:

$$\log W = \log \overline{W} + m_i \frac{\partial}{\partial n_i} \log \overline{W} + \frac{m_i^2}{2!} \frac{\partial^2}{\partial n_i^2} \log W + \ldots . \quad (28)$$

Formula (25) gives the value of the derivative of $\log W$ with respect to n_i; we may now suppose that N remains constant, and we may write

$$\frac{\partial}{\partial n_i} (\log W) = \log \frac{N}{G - aN} \frac{g_i - a\bar{n}_i}{\bar{n}_i} - pE_i = 0, \quad . \quad (29)$$

$$\frac{\partial^2}{\partial n_i^2} (\log W) = - \frac{g_i}{\bar{n}_i (g_i - a\bar{n}_i)}.$$

This reasoning holds for all values of the constant a; the case of radiation corresponds to the assumption $a = -1$.

Formula (28) then gives:

$$W = \overline{W} e^{- \frac{g_i}{\bar{n}_i (g_i - a\bar{n}_i)} \frac{m_i^2}{2}}. \quad . \quad . \quad . \quad . \quad (30)$$

It follows that the mean square of the fluctuations m_i is equal to

$$\overline{m_i}^2 = \bar{n}_i \left(1 - a \frac{\bar{n}_i}{g_i} \right). \quad . \quad . \quad . \quad . \quad . \quad (31)$$

When the projectiles are independent of each other ($a = 0$) we obtain the classical result; but the assumption of coupling of the photons ($a = -1$) automatically leads to Einstein's formula for the fluctuations.

6. **Discussion; Comparison with other Deductions of Planck's Formula.**—The above reasoning shows that coupling of the photons must be assumed to exist if we are to obtain the correct laws of black radiation.* The special feature of the theory which I have developed is its clear statement of the conditions under which this association of quanta takes place.

In order to carry the discussion further, we must remember the simple connexion between the evaluation of the number of compartments (formula (5*b*)) and the enumeration of the modes of vibration of an empty enclosure; the latter was given by Jeans in a form which has become classical. On investigating the number of proper modes

* This point has previously been noted since the rise of the quantum theory; Planck assumes that a resonator can have energies $h\nu$, $2h\nu$, . . . $ph\nu$, . . .; Jeans ("Report on Radiation", *Phys. Soc.*, London, 1916) is obliged to make a similar assumption. Louis de Broglie (*Ann. de Phys.*, Vol. **3** (1925), p. 86, Chapter VII) also dwells on the necessity of associating light quanta into groups with the same phase wave.

of vibration of an enclosure of volume V, he found that there are

$$d\mathcal{M} = V\frac{8\pi\nu^2}{c^3}\Delta\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

with frequencies lying between ν and $\nu + \Delta\nu$. This number is therefore equal to the number of compartments in the same interval (see formula (5*b*)), and it is natural to assume that each compartment corresponds to one mode of vibration.

Now consider an atom placed in the enclosure V; for simplicity we shall suppose that this atom possesses only two levels of energy e_1 and e_2, with

$$e_1 - e_2 = h\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

In a noteworthy memoir,* Einstein has investigated the conditions of absorption and emission which allow the atom to be in equilibrium with the radiation in the enclosure. With regard to this, he stated the following laws:

$$\left\{\begin{array}{ll}\text{Probability of absorption of } h\nu & B\rho_\nu \\ \text{Probability of emission of } h\nu & A + B\rho_\nu\end{array}\right\}, \quad (34)$$

with

$$\frac{A}{B} = \frac{8\pi h\nu^3}{c^3},$$

where $\rho_\nu\Delta\nu$ is the mean density of the energy of radiation of frequency $(\nu, \nu + \Delta\nu)$ in the enclosure.

The origin of the numerical value of the ratio $\frac{A}{B}$ in this formula appears somewhat mysterious. We shall show that these laws are identical with those we assumed in the last section.

Instead of ρ_ν, the energy of the radiation, let us introduce u_ν, the value of the average energy of one of the resonators of radiation. Using the enumeration (formula (32)) we shall have

$$\rho_\nu\Delta\nu = \frac{8\pi\nu^2}{c^3}u_\nu\Delta\nu = \frac{8\pi h\nu^3}{c^3}\bar{p}_\nu\Delta\nu, \quad u_\nu = \bar{p}_\nu h\nu, \quad . \quad (35)$$

where $\bar{p}_\nu$ is the average number of quanta $h\nu$ per resonator ν.

Comparing the formulæ (34) and (35), we see that Einstein's laws may be put in the form

$$\left\{\begin{array}{l}\text{Probability of emission} = A(1 + \bar{p}_\nu) \\ \text{Probability of absorption} = A\bar{p}_\nu\end{array}\right\}, \quad . \quad . \quad (36)$$

whence we obtain the following statements:

* A. Einstein, *Phys. Zeits.*, Vol. 18 (1917), p. 121. See also L. Brillouin, "La théorie des quanta et l'atome de Bohr" (*Conférences-rapports*, Paris, 1922, Chapter VI).

I. At a given instant, the atom can exchange a quantum $h\nu$ with only one resonator of radiation, or—which comes to the same—with only one of the h^3 compartments corresponding to the radiation ν.

II. The probability that the atom should emit a quantum $h\nu$ is proportional to $1 + p$, where p is the number of quanta already in the compartment h^3.

III. The probability that the atom should absorb a quantum $h\nu$ is proportional to the number p.

It is very curious to find that the statements I and II are identical with those which I was led to formulate in § 3. Thus there is complete agreement between the point of view which I have developed and the laws of emission and absorption stated by Einstein.

This example shows that circumstances enable us to identify each projectile (photon or atom) and to distinguish it from its fellows. The complete identity assumed by Bose is a fiction which cannot be realized: each of the objects has followed a different route, and this enables us to distinguish it from the others. The photon emitted at a given instant comes from a determinate atom, and the other photons which had accumulated in the same compartment had previously had quite distinct histories.

Whenever we wish to make the mechanism of any elementary phenomenon clear, we are under the conditions which I assumed as the logical basis of statistics. We imagine that a given atom is observed, the motion of which is disturbed by collision with a photon or another special atom. The circumstances of the collision are then influenced not only by the initial conditions, but by the nature of the final state that the collision is capable of causing; for the probability that a new atom (or photon) should result, with its representative point within a certain compartment h^3, depends on p, the number of representative points already situated in this compartment; according to our assumptions this probability is proportional to $1 - pa$, with $a = 1$ (Pauli); $a = 0$ (classical); or $a = -1$ (Bose-Einstein).

Let us investigate the probability of a transformation in which a certain number n_1 of particles of the first type (distinguished by a value a_1 for a) change their compartments, while n_2 particles of the second type (a_2) are shifted in some other way. Let $p_1, p_1', \ldots, p_1^{(n_1)}$ be the numbers of particles of the first type in their n_1 initial compartments, and let $P_1, P_1', \ldots, P_1^{(n_1)}$ be the numbers of particles in the final compartments after the transformation. Let p_2^i and P_2^i be the corresponding numbers of particles of the second type in the initial and final compartments. The probability of the transformation will be proportional to the numbers of particles situated in the initial compartments and to the expressions $1 - Pa$ relating to the final

compartments; we therefore obtain the velocity of transformation

$$A \prod_{i,k} p_1^i p_2^k (1 - P_1^i a_1)(1 - P_2^k a_2). \quad . \quad . \quad (37)$$

Expressions of this type have already been proposed by various writers in a series of particular cases.* Instead of connecting them with a general hypothesis, however, they obtain them in every case by investigating the type of laws capable of yielding Planck's formula for radiation and the classical formula for the distribution of the velocities of the molecules of a gas.

* W. Pauli, *Zts. f. Phys.*, Vol. **18** (1923), p. 272, and Vol. **22** (1924), p. 261; L. S. Ornstein and H. C. Burger, *Zts. f. Phys.*, Vol. **20** (1923), pp. 345, 351; A. Einstein and P. Ehrenfest, *Zts. f. Phys.*, Vol. **19** (1923), p. 301; Jordan, *Zts. f. Phys.*, Vol. **33** (1925), p. 649; L. de Broglie, *Ondes et mouvements* (Gauthier-Villars, Paris, 1926), Chapter XIV.

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